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Five-Brane Superpotentials in Heterotic M -theory

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ABSTRACT

The open supermembrane contribution to the non-perturbative superpotential of bulk space five-branes in heterotic M -theory is presented. We explicitly compute the superpotential for the modulus associated with the separation of a bulk five-brane from an end-of-the-world three-brane. The gauge and κ -invariant boundary strings of such open supermembranes are given and the role of the holomorphic vector bundle on the orbifold fixed plane boundary is discussed in detail.

1 Introduction

One of the principal problems obstructing attempts to obtain the standard model from M -theory, that is, how to obtain chiral fermions in the low energy effective theory, was solved in several papers by Hořava-Witten [1, 2] and Witten [3]. These authors compactified eleven-dimensional $N = 1$ supergravity on the orbifold S^1/\mathbb{Z}_2 , producing chiral fermions on each of the two ten-dimensional orbifold fixed planes. They then showed that cancellation of the gravitational anomalies induced by these chiral fermions uniquely requires that each orbifold fixed plane supports an $N = 1$, E_8 super-Yang-Mills multiplet. Hořava-Witten theory, therefore, is a theory with an eleven-dimensional “bulk” space bounded on two sides by ten-dimensional S^1/\mathbb{Z}_2 orbifold planes. The relation of this theory to $N = 1$ four-dimensional theories was explored in [3, 4, 5], but these papers compactified Hořava-Witten theory directly to four-dimensions.

In a series of papers [6]–[9], it was shown that it is natural to compactify Hořava-Witten theory, not directly to four-dimensions, but, rather, on a Calabi-Yau threefold to a theory with a five-dimensional bulk space bounded, on each end of the orbifold, by four-dimensional BPS three-branes. In this compactification, called heterotic M -theory, our observable world arises as the worldvolume theory of one of the boundary three-branes, the other boundary brane forming a hidden sector. Heterotic M -theory represents a fundamental realization of a “brane universe” directly from M -theory. It was demonstrated explicitly in [10]–[14] that both grand unified theories of particle physics and the standard model can arise on the observable three-brane by appropriately specifying semi-stable, holomorphic E_8 vector bundles on the associated Calabi-Yau space. These Yang-Mills “instantons” break E_8 to phenomenologically interesting gauge groups, such as $SU(3) \times SU(2) \times U(1)$, and lead to three families of quarks and leptons. However, it was shown in these papers that, generically, anomaly cancellation requires the existence of BPS five-branes, wrapped on holomorphic curves in the Calabi-Yau threefold, in the bulk space. These five-branes represent new, non-perturbative physics that might have dramatic effects both in low energy particle physics and in cosmology [15]–[24]. Therefore, it is of importance to have a detailed understanding of their dynamics.

With this in mind, in a recent paper [25], we computed the superpotential induced at low energy by the exchange of open supermembranes between the two orbifold fixed planes. This superpotential is an explicit holomorphic function of the $(1,1)$ -moduli of the Calabi-Yau threefold. In addition, we showed that this superpotential is only non-vanishing under restrictive topological conditions on the end-of-the-world orbifold plane

instantons, namely, that the restriction of each vector bundle to the holomorphic curve around which the supermembrane is wrapped be trivial. In this paper, we extend these results to compute the low energy $N = 1$ superpotential induced by the exchange of open supermembranes between one end-of-the-world BPS three-brane and a wrapped five-brane in the bulk space. This calculation, although related to the one performed for the two orbifold fixed planes, has many features that are unique to the bulk space five-brane. We find that the superpotential is a holomorphic function of a new, composite modulus. This modulus is a specific combination of the translation modulus of the five-brane, the real and imaginary parts of the $(1, 1)$ -modulus associated with the holomorphic curve on which the five-brane is wrapped and the “axion” modulus, which is related to the worldvolume two-form of the five-brane. Again, we find that this superpotential is only non-vanishing if the vector bundle associated with the end-of-the-world three-brane is trivial when restricted to the holomorphic curve on which the five-brane and open supermembrane are wrapped.

Specifically, we do the following. In Section 2 we discuss supermembranes and five-branes in both eleven-dimensional supergravity and in Hořava-Witten theory. The κ -invariant action for an open supermembrane with one boundary string on an orbifold fixed plane and the other on a bulk space five-brane is studied in detail in Section 3. Section 4 is devoted to a discussion of the compactification of this theory on a Calabi-Yau threefold to heterotic M -theory and the further dimensional reduction on the S^1/\mathbb{Z}_2 orbifold. The effective action is shown to reduce to that of the heterotic superstring coupled to one E_8 gauge background, a Neveu-Schwarz five-brane and wrapped on a holomorphic curve in the Calabi-Yau manifold. In Section 5, we review the relevant moduli in heterotic M -theory and their reduction to the four-dimensional effective theory. We discuss in detail the $(1, 1)$ -modulus \mathcal{T} associated with the holomorphic curve and the translational chiral multiplet \mathbf{Y} that will appear in the superpotential. We also discuss the method for calculating the superpotential from the open supermembrane contribution to the relevant fermion two-point function. Section 6 is devoted to the explicit calculation of the non-perturbative corrections to this two-point function using a saddle point approximation. We present a careful discussion of gauge fixing and zero-modes, calculate the bosonic holomorphic contribution and compute the formal expressions for the determinants associated with quadratic fluctuation terms. In Section 7 we calculate the Wess-Zumino-Witten determinant related to quadratic fluctuations in the background of an E_8 gauge instanton. It is shown that this determinant is only non-vanishing if the restriction of the holomorphic vector bundle to the curve on which the heterotic string is wrapped is trivial. Finally, we extract the complete expression for the

superpotential associated with the five-brane translation modulus in Section 8. Our notation and conventions are discussed in the Appendix.

Our work, both in this paper and in [25], is based on the ground-breaking formalism presented in [26, 27]. Recently, a paper due to Moore, Peradze and Saulina [29] appeared which studied topics similar to those presented here and in [25]. Some of our results are similar to theirs and much is new or complementary. We acknowledge their work and appreciate their pre-announcement of our independent study of this subject. We want to point out and emphasize the paper of Derendinger and Sauser [28] on the perturbative low energy effective theory of five-branes in heterotic M -theory. These authors elucidated the relevant moduli associated with five-brane dynamics and computed their contribution to the four-dimensional Kähler potential. In this paper, we add the non-perturbative superpotential contributions. We note that the same moduli naturally arise in our calculation, in a very different context.

2 Membranes and Five-Branes in Hořava-Witten Theory:

Eleven-Dimensional Supergravity, Membranes and Five-Branes:

$N = 1$ supersymmetry in eleven-dimensions has 32 supercharges and consists of a single supergravity multiplet [30] containing as its component fields a graviton $\hat{g}_{\hat{M}\hat{N}}$, a three-form $\hat{C}_{\hat{M}\hat{N}\hat{P}}$ and a Majorana gravitino $\hat{\Psi}_{\hat{M}}$. The field strength of the three-form, defined by $\hat{G} = d\hat{C}$, has as its components $\hat{G}_{\hat{M}\hat{N}\hat{P}\hat{Q}} = 24\partial_{[\hat{M}}\hat{C}_{\hat{N}\hat{P}\hat{Q}]}$. We denote the coordinates of the real eleven manifold M_{11} as $(\hat{x}^{\hat{0}}, \dots, \hat{x}^{\hat{9}}, \hat{x}^{\hat{11}})$. The associated action is invariant under the supersymmetry transformations of the component fields. For our purposes, we need only specify the supersymmetry variation of the gravitino field $\hat{\Psi}_{\hat{M}}$, which is given by

$$\delta_{\hat{\varepsilon}}\hat{\Psi}_{\hat{M}} = \hat{D}_{\hat{M}}\hat{\varepsilon} + \frac{\sqrt{2}}{288}(\hat{\Gamma}_{\hat{M}}^{\hat{N}\hat{P}\hat{Q}\hat{R}} - 8\delta_{\hat{M}}^{\hat{N}}\hat{\Gamma}^{\hat{P}\hat{Q}\hat{R}})\hat{\varepsilon}\hat{G}_{\hat{N}\hat{P}\hat{Q}\hat{R}} + \dots, \quad (2.1)$$

where $\hat{\varepsilon}$ is the Majorana supersymmetry parameter and the dots denote terms that involve the fermion fields of the theory. The eleven-dimensional spacetime Dirac matrices $\hat{\Gamma}_{\hat{M}}$ satisfy $\{\hat{\Gamma}_{\hat{M}}, \hat{\Gamma}_{\hat{N}}\} = 2\hat{g}_{\hat{M}\hat{N}}$. $N = 1$ eleven-dimensional supergravity can be formulated in a superspace with coordinates

$$\hat{z}^{\mathbb{M}} = (\hat{x}^{\hat{M}}, \hat{\theta}^{\hat{\mu}}), \quad (2.2)$$

where $\hat{x}^{\hat{M}}$, $\hat{M} = 0, \dots, 9, 11$ are the bosonic coordinates introduced above and $\hat{\theta}^{\hat{\mu}}$, $\hat{\mu} = 1, \dots, 32$ are anti-commuting coordinates in a thirty-two component Majorana spinor. In

this formulation, the graviton and three-form appear as the lowest components of the super-elfbein $\hat{\mathbb{E}}_{\hat{\mathbb{M}}}^{\hat{\mathbb{A}}}$ and the super-three-form $\hat{\mathbb{C}}_{\hat{\mathbb{C}}\hat{\mathbb{B}}\hat{\mathbb{A}}}$ respectively. The gravitino arises at order $\hat{\theta}$ in the expansion of $\hat{\mathbb{E}}_{\hat{\mathbb{M}}}^{\hat{\mathbb{A}}}$.

It is well-known that there is a 2 + 1-dimensional “electrically charged” membrane solution of the M -theory equations of motion that preserves one-half of the supersymmetries [31], that is, 16 supercharges. The worldvolume action for this supermembrane coupled to background eleven-dimensional supergravity is known [32]. It is given, in the target superspace formulation, by

$$S_{SM} = -T_M \int_{\Sigma} d^3\hat{\sigma} (\sqrt{-\det \hat{g}_{ij}} - \frac{1}{6} \hat{\varepsilon}^{ijk} \hat{\Pi}_i^{\hat{\mathbb{A}}} \hat{\Pi}_j^{\hat{\mathbb{B}}} \hat{\Pi}_k^{\hat{\mathbb{C}}} \hat{\mathbb{C}}_{\hat{\mathbb{C}}\hat{\mathbb{B}}\hat{\mathbb{A}}}), \quad (2.3)$$

where

$$T_M = (2\pi^2/\kappa^2)^{1/3} \quad (2.4)$$

is the membrane tension of mass dimension three and $\hat{\sigma}^{\hat{i}}, \hat{i} = 0, 1, 2$ are the intrinsic coordinates of the membrane worldvolume Σ . Parameter κ is the eleven-dimensional Newton constant. Furthermore,

$$\hat{g}_{ij} = \hat{\Pi}_i^{\hat{A}} \hat{\Pi}_j^{\hat{B}} \eta_{\hat{A}\hat{B}}, \quad \hat{\Pi}_i^{\hat{\mathbb{A}}} = \partial_i \hat{\mathbb{Z}}^{\hat{\mathbb{M}}} \hat{\mathbb{E}}_{\hat{\mathbb{M}}}^{\hat{\mathbb{A}}}, \quad (2.5)$$

where $\hat{\mathbb{Z}}^{\hat{\mathbb{M}}}$ represents the superembedding $\hat{\mathbb{Z}} : \Sigma^{3|0} \rightarrow M^{11|32}$, whose bosonic and fermionic component fields are the background coordinates

$$\hat{\mathbb{Z}}^{\hat{\mathbb{M}}}(\hat{\sigma}) = (\hat{X}^{\hat{M}}(\hat{\sigma}), \hat{\Theta}^{\hat{\mu}}(\hat{\sigma})), \quad (2.6)$$

respectively. The action is a sigma-model since the super-elfbeins $\hat{\mathbb{E}}_{\hat{\mathbb{M}}}^{\hat{\mathbb{A}}}$ and the super-three-form $\hat{\mathbb{C}}_{\hat{\mathbb{C}}\hat{\mathbb{B}}\hat{\mathbb{A}}}$ both depend on the superfields $\hat{\mathbb{Z}}^{\hat{\mathbb{M}}}$. The super-elfbeins have, as their first bosonic and fermionic component in the $\hat{\Theta}$ expansion, the bosonic elfbeins $\hat{E}_{\hat{M}}^{\hat{A}}$ and the gravitino $\hat{\Psi}_{\hat{M}}^{\hat{\alpha}}$ respectively, while the super-three-form has the bosonic three-form from eleven-dimensional supergravity as its leading field component.

The fact that the membrane solution of M -theory preserves one-half of the supersymmetries translates, when speaking in supermembrane worldvolume language, into the fact that the action (2.3) exhibits a local fermionic invariance, κ -invariance, that is used to gauge away half of the fermionic degrees of freedom. Specifically, the supermembrane action is invariant under the local fermionic symmetries

$$\delta_{\hat{\kappa}} \hat{\Theta} = 2\hat{P}_+ \hat{\kappa} + \dots, \quad \delta_{\hat{\kappa}} \hat{X}^{\hat{M}} = 2\hat{\bar{\Theta}} \hat{\Gamma}^{\hat{M}} \hat{P}_+ \hat{\kappa} + \dots, \quad (2.7)$$

where $\hat{\kappa}(\hat{\sigma})$ is an eleven-dimensional local spinor parameter and \hat{P}_{\pm} are the projection operators

$$\hat{P}_{\pm} \equiv \frac{1}{2} (1 \pm \frac{1}{6\sqrt{-\det \hat{g}_{ij}}} \hat{\varepsilon}^{ijk} \hat{\Pi}_i^{\hat{A}} \hat{\Pi}_j^{\hat{B}} \hat{\Pi}_k^{\hat{C}} \hat{\Gamma}_{\hat{A}\hat{B}\hat{C}}), \quad (2.8)$$

obeying

$$\hat{P}_\pm^2 = \hat{P}_\pm, \quad \hat{P}_+ \hat{P}_- = 0, \quad \hat{P}_+ + \hat{P}_- = 1. \quad (2.9)$$

It follows from the first equation in (2.7) that the $\hat{P}_+ \hat{\Theta}$ component of spinor $\hat{\Theta}$ can be transformed away by a κ -transformation. Note that (2.7) includes only the leading order terms in $\hat{\Theta}$, which is all that is required to discuss the supersymmetry properties of the membrane. It can be shown that the membrane action (2.3) will be invariant under transformations (2.7) if and only if the background superfields $\hat{\mathbb{E}}_{\hat{\mathbb{M}}}^{\hat{\mathbb{A}}}$ and $\hat{\mathbb{C}}_{\hat{\mathbb{C}}\hat{\mathbb{B}}\hat{\mathbb{A}}}$ satisfy the eleven-dimensional supergravity constraint equations. However, the general bosonic membrane configuration $\hat{X}(\hat{\sigma})$ is not invariant under global supersymmetry transformations

$$\delta_{\hat{\varepsilon}} \hat{\Theta} = \hat{\varepsilon}, \quad \delta_{\hat{\varepsilon}} \hat{X}^{\hat{M}} = \hat{\varepsilon} \hat{\Gamma}^{\hat{M}} \hat{\Theta}, \quad (2.10)$$

where $\hat{\varepsilon}$ is an eleven-dimensional spinor independent of $\hat{\sigma}$. Nevertheless, one-half of the supersymmetries will remain unbroken if and only if (2.10) can be compensated for by a κ -transformation with a suitable parameter $\hat{\kappa}(\hat{\sigma})$. That is

$$\begin{aligned} \delta \hat{\Theta} &= \delta_{\hat{\varepsilon}} \hat{\Theta} + \delta_{\hat{\kappa}} \hat{\Theta} \\ &= \hat{\varepsilon} + 2 \hat{P}_+ \hat{\kappa}(\hat{\sigma}) = 0. \end{aligned} \quad (2.11)$$

In order for this to be satisfied, a necessary condition is that

$$\hat{P}_- \hat{\varepsilon} = 0. \quad (2.12)$$

In addition to the supermembrane, it is well-known that there is a six-dimensional “solitonic” five-brane solution of M -theory that preserves one-half of the supersymmetries [1], that is, 16 supercharges. The worldvolume action for this five-brane coupled to background eleven-dimensional supergravity is known [33], but it is not necessary to give its explicit form in this paper. Here, it suffices to note the following. The scale of the action is set by T_5 , which is the five-brane tension of mass dimension six given by

$$T_5 = (4\pi/\kappa^4)^{1/3}. \quad (2.13)$$

It follows from (2.4) that the relation between T_5 and T_M is

$$T_5 = \frac{2\pi}{\kappa^2} \frac{1}{T_M}. \quad (2.14)$$

The five-brane worldvolume \underline{M}_6 has the six intrinsic coordinates $\xi^{\underline{r}}, \underline{r} = 0, \dots, 5$ and world-volume metric

$$\underline{g}_{\underline{rs}} = \hat{\Pi}_{\underline{r}}^{\hat{A}} \hat{\Pi}_{\underline{s}}^{\hat{B}} \eta_{\hat{A}\hat{B}}, \quad \hat{\Pi}_{\underline{r}}^{\hat{A}} = \partial_{\underline{r}} \hat{\mathbb{Y}}^{\hat{\mathbb{M}}} \hat{\mathbb{E}}_{\hat{\mathbb{M}}}^{\hat{A}}, \quad (2.15)$$

where $\underline{\hat{Y}}^{\hat{M}}$ represents the superembedding $\hat{Y} : M_6^{6|0} \rightarrow M^{11|32}$ with

$$\underline{\hat{Y}}^{\hat{M}}(\xi) = (\underline{\hat{Y}}^{\hat{M}}(\xi), \underline{\hat{\Xi}}^{\hat{\mu}}(\xi)). \quad (2.16)$$

In addition to $\underline{\hat{Y}}^{\hat{M}}$ and $\underline{\hat{\Xi}}^{\hat{\mu}}$, the five-brane theory also requires the introduction of a world-volume two-form, $\underline{D}_{rs}(\xi)$, whose field strength is anti-self-dual. Finally, we note that the five-brane action contains explicit couplings to the super-elfbeins $\hat{\mathbb{E}}_{\hat{M}}^{\hat{A}}$ and super-three-form $\hat{\mathbb{C}}_{\hat{C}\hat{B}\hat{A}}$ of the eleven-dimensional background supergravity.

As for the supermembrane, the five-brane worldvolume action exhibits a κ -invariance that can be used to gauge away half of the fermionic degrees of freedom. Specifically, the action is invariant under

$$\delta_{\hat{\kappa}} \underline{\hat{\Xi}} = 2\mathcal{P}_{+\hat{\kappa}} + \dots, \quad \delta_{\hat{\kappa}} \underline{\hat{Y}}^{\hat{M}} = 2\underline{\hat{\Xi}} \hat{\Gamma}^{\hat{M}} \mathcal{P}_{+\hat{\kappa}} + \dots, \quad \delta_{\hat{\kappa}} \underline{D}_{rs} = 2\underline{\mathbb{C}}_{rs\hat{\mu}} \mathcal{P}_{+\hat{\kappa}}^{\hat{\mu}}, \quad (2.17)$$

where $\hat{\kappa}(\xi)$ is an eleven-dimensional local spinor parameter,

$$\underline{\mathbb{C}}_{rs\hat{\mu}} = \underline{\hat{\Pi}}_{\underline{r}}^{\hat{A}} \underline{\hat{\Pi}}_{\underline{s}}^{\hat{B}} \hat{\mathbb{C}}_{\hat{\mu}\hat{B}\hat{A}} \quad (2.18)$$

and \mathcal{P}_{\pm} are projection operators. In general, these operators depend in a complicated way on the three-form

$$\underline{\mathbb{H}}_{rst} = (d\underline{D})_{rst} - \underline{\mathbb{C}}_{rst}, \quad (2.19)$$

where

$$\underline{\mathbb{C}}_{rst} = \underline{\hat{\Pi}}_{\underline{r}}^{\hat{A}} \underline{\hat{\Pi}}_{\underline{s}}^{\hat{B}} \underline{\hat{\Pi}}_{\underline{t}}^{\hat{C}} \hat{\mathbb{C}}_{\hat{C}\hat{B}\hat{A}} \quad (2.20)$$

is the pullback of the supergravity super-three-form onto the five-brane worldvolume. If, however, one chooses

$$\underline{\mathbb{H}}_{rst} = 0, \quad (2.21)$$

which we will do for the remainder of this paper, then these projection operators simplify and are given by

$$\mathcal{P}_{\pm} = \frac{1}{2} \left(1 \pm \frac{1}{6! \sqrt{-\det g_{rs}}} \varepsilon^{r_1 \dots r_6} \underline{\hat{\Pi}}_{r_1}^{\hat{A}_1} \dots \underline{\hat{\Pi}}_{r_6}^{\hat{A}_6} \hat{\Gamma}_{\hat{A}_1 \dots \hat{A}_6} \right). \quad (2.22)$$

Note that (2.17) includes only the leading order terms in $\underline{\hat{\Xi}}$, which is all that is required to discuss the supersymmetry properties of the five-brane. It can be shown that the five-brane action will be invariant under κ -transformation (2.17) if and only if the background superfields $\hat{\mathbb{E}}_{\hat{M}}^{\hat{A}}$ and $\hat{\mathbb{C}}_{\hat{C}\hat{B}\hat{A}}$ satisfy the constraint equations of eleven-dimensional supergravity. Using a κ -transformation with a suitable parameter $\hat{\kappa}(\xi)$, the global supersymmetry transformations

$$\delta_{\hat{\kappa}} \underline{\hat{\Xi}} = \hat{\kappa}, \quad \delta_{\hat{\kappa}} \underline{\hat{Y}}^{\hat{M}} = \underline{\hat{\Xi}} \hat{\Gamma}^{\hat{M}} \hat{\kappa}, \quad (2.23)$$

where $\hat{\underline{\epsilon}}$ is an eleven-dimensional spinor independent of ξ , can be compensated for to leave one-half of the supersymmetries unbroken. That is

$$\delta\hat{\underline{\epsilon}} = \hat{\underline{\epsilon}} + 2\mathcal{P}_+\hat{\underline{\kappa}}(\xi) = 0. \quad (2.24)$$

For this to be satisfied, a necessary condition is that

$$\mathcal{P}_-\hat{\underline{\epsilon}} = 0. \quad (2.25)$$

It is well-known that after fixing this κ -gauge, the 16 unbroken supercharges arrange themselves as a six-dimensional (2,0) supersymmetry and that the worldvolume theory of the five-brane consists of a single tensor multiplet. This supermultiplet contains as its component fields

$$(\underline{D}_{rs}, \hat{\underline{Y}}^{\hat{p}}, \chi), \quad \hat{p} = 6, \dots, 9, 11, \quad (2.26)$$

where the field strength of \underline{D}_{rs} is anti-self-dual, the five scalars $\hat{\underline{Y}}^{\hat{p}}$ label the transverse translational modes of the five-brane and χ are the associated fermions.

We now turn to a discussion of supermembranes and five-branes in Hořava-Witten theory.

Five-Branes and Membranes in Hořava-Witten Theory:

When M -theory is compactified on S^1/\mathbb{Z}_2 , it describes the low energy limit of the strongly coupled heterotic string theory [1, 2]. We choose $\hat{x}^{\hat{1}1}$ as the orbifold direction and parametrize S^1 by $\hat{x}^{\hat{1}1} \in [-\pi\rho, \pi\rho]$ with the endpoints identified. The \mathbb{Z}_2 symmetry acts by further identifying any point $\hat{x}^{\hat{1}1}$ with $-\hat{x}^{\hat{1}1}$ and, therefore, gives rise to two ten-dimensional fixed hyperplanes at $\hat{x}^{\hat{1}1} = 0$ and $\hat{x}^{\hat{1}1} = \pi\rho$. We will denote the manifold of either of them by M_{10} . Since, at each \mathbb{Z}_2 hyperplane, only the field components that are even under the \mathbb{Z}_2 action can survive, the eleven-dimensional supergravity in the bulk space is projected into $N = 1$ ten-dimensional chiral supergravity on each boundary. $N = 1$ supersymmetry in ten-dimensions preserves 16 supercharges. Furthermore, cancellation of the chiral anomaly in this theory requires the existence of an $N = 1$, E_8 super-Yang-Mills multiplet on each fixed hyperplane [1, 2]. Therefore, the effective action for M -theory on S^1/\mathbb{Z}_2 describes the coupling of two ten-dimensional E_8 super-Yang-Mills theories, one on each hyperplane, to eleven-dimensional supergravity in the bulk space. In order to cancel all chiral anomalies on the hyperplanes, the action has to be supplemented by the modified Bianchi identity¹

$$(d\hat{G})_{\hat{1}1MNPQ} = -\frac{1}{4\pi} \left(\frac{\kappa}{4\pi} \right)^{2/3} \left(J^{(1)}\delta(\hat{x}^{\hat{1}1}) + J^{(2)}\delta(\hat{x}^{\hat{1}1} - \pi\rho) \right)_{MNPQ}, \quad (2.27)$$

¹The normalization of \hat{G} adopted here differs from [1] by a factor of $\sqrt{2}$ but it agrees with [35], in which one considers, as we will do in this paper, the superfield version of the Bianchi identities.

where

$$J^{(n)} = \text{tr} F^{(n)} \wedge F^{(n)} - \frac{1}{2} \text{tr} R \wedge R, \quad (2.28)$$

for $n = 1, 2$ and $M, N = 0, \dots, 9$. The solutions to the equations of motion must respect the \mathbb{Z}_2 orbifold symmetry. For the purposes of this paper, we need only specify the transformation property of the gravitino under \mathbb{Z}_2 . This is given by

$$\hat{\Psi}_M(\hat{x}^{\hat{1}\hat{1}}) = \hat{\Gamma}_{\hat{1}\hat{1}} \hat{\Psi}_M(-\hat{x}^{\hat{1}\hat{1}}), \quad \hat{\Psi}_{\hat{1}\hat{1}}(\hat{x}^{\hat{1}\hat{1}}) = -\hat{\Gamma}_{\hat{1}\hat{1}} \hat{\Psi}_{\hat{1}\hat{1}}(-\hat{x}^{\hat{1}\hat{1}}). \quad (2.29)$$

where $\hat{\Gamma}_{\hat{1}\hat{1}} = \hat{\Gamma}_{\hat{0}} \hat{\Gamma}_{\hat{1}} \cdots \hat{\Gamma}_{\hat{9}}$. In order for the supersymmetry transformations of the gravitino to be consistent with the \mathbb{Z}_2 symmetry, the eleven-dimensional Majorana spinor $\hat{\varepsilon}$ in (2.1) must satisfy

$$\hat{\varepsilon}(\hat{x}^{\hat{1}\hat{1}}) = \hat{\Gamma}_{\hat{1}\hat{1}} \hat{\varepsilon}(-\hat{x}^{\hat{1}\hat{1}}). \quad (2.30)$$

This equation does not restrict the number of independent components of the spinor fields $\hat{\varepsilon}$ at any point in the bulk space. However, at each of the \mathbb{Z}_2 hyperplanes, constraint (2.30) becomes the ten-dimensional chirality condition

$$\frac{1}{2}(1 - \hat{\Gamma}_{\hat{1}\hat{1}})\hat{\varepsilon} = 0, \quad \text{at } \hat{x}^{\hat{1}\hat{1}} = 0, \pi\rho. \quad (2.31)$$

This condition implies that the theory exhibits $N = 1$ supersymmetry on each of the ten-dimensional orbifold fixed planes.

Five-brane solutions were explicitly constructed for Hořava-Witten theory in [36]. There, the five-brane solution of eleven-dimensional supergravity was shown to satisfy the equations of motion of the theory subject to the \mathbb{Z}_2 constraints. There are two different ways to orient the five-brane with respect to the orbifold direction, that is, \hat{x}^{11} can be either a transverse coordinate or a coordinate oriented in the direction of the five-brane. In the first case, the five-brane is parallel to the hyperplanes. In the second case, it intersects each of them along a 4+1 dimensional brane. Note, however, that there is no BPS 4-brane in ten-dimensional heterotic string theory. It follows that the second orientation cannot preserve supercharges compatible with the $N = 1$ supersymmetry on the boundary hyperplanes. Hence, such five-branes are not of interest from the point of view of this paper, and we consider only five-branes parallel to the orbifold fixed planes. We now show that these parallel five-branes do conserve supercharges compatible with the boundary hyperplane supersymmetry. As shown above, for any orientation of the five-brane, one half of the supersymmetries will remain unbroken if and only if the target supersymmetry transformation with spinor parameter $\hat{\underline{\varepsilon}}$ can be compensated for by a κ -transformation with a suitable parameter $\hat{\underline{\kappa}}(\xi)$. For this to be the case $\hat{\underline{\varepsilon}}$ must satisfy $\mathcal{P}_- \hat{\underline{\varepsilon}} = 0$, where \mathcal{P}_- is given in (2.22). Now choose the orientation

of the fivebrane to be parallel to the orbifold fixed planes. In this case, we can take the fields $\underline{\hat{Y}}$ and $\underline{\hat{\Xi}}$ such that

$$\begin{aligned}\underline{\hat{Y}}^{\hat{p}'} &= \delta_{\underline{r}}^{\hat{p}'} \xi^{\underline{r}}, & \hat{p}', \underline{r} &= 0, \dots, 5, \\ \underline{\hat{Y}}^{\hat{p}} &= 0, & \hat{p} &= 6, \dots, 9, \\ \underline{\hat{Y}}^{\hat{1}1} &= Y, & \underline{\hat{\Xi}} &= 0,\end{aligned}\tag{2.32}$$

where Y is the location of the five-brane along the orbifold direction. Expression (2.25) then becomes

$$\mathcal{P}_{-\underline{\hat{\Xi}}} = \frac{1}{2}(1 - \hat{\Gamma}_{012345})\underline{\hat{\Xi}} = 0,\tag{2.33}$$

where $\hat{\Gamma}_{012345} = \hat{\Gamma}_{\hat{0}} \cdots \hat{\Gamma}_{\hat{5}}$. Note that the 16 supercharges preserved on the five-brane world-volume by chirality condition (2.33) form a (2,0)-supersymmetry on the five-brane.

As discussed previously, the five-brane worldvolume fields form a tensor supermultiplet of (2,0)-supersymmetry. This contains, among other things, a two-form \underline{D}_{rs} whose field strength is anti-self-dual. It is important to note that the presence of an anti-self-dual tensor in six-dimensions leads to a gravitational anomaly on the five-brane worldvolume. This must be canceled by adding the appropriate higher dimensional interactions to the eleven-dimensional supergravity theory and by modifying Bianchi identity (2.27) to

$$(d\hat{G})_{\hat{1}LMNPQ} = -\frac{1}{4\pi} \left(\frac{\kappa}{4\pi} \right)^{2/3} \left(J^{(1)} \delta(\hat{x}^{\hat{1}1}) + J^{(2)} \delta(\hat{x}^{\hat{1}1} - \pi\rho) + J_5 \delta(\hat{x}^{\hat{1}1} - Y) \right)_{MNPQ},\tag{2.34}$$

where J_5 is the four-form source that is Poincare dual to the homology class of the complex curve on which the five-brane is wrapped.

Now consider supermembranes in the Hořava-Witten context. We begin by assuming there are no five-branes in the bulk space. There are two different ways to orient the membrane with respect to the orbifold direction, that is, $\hat{x}^{\hat{1}1}$ can either be a transverse coordinate or a coordinate oriented in the direction of the membrane. In the first case, the membrane is parallel to the hyperplanes. In the second case, it extends between the two hyperplanes and intersects each of them along a 1 + 1-dimensional string. This latter configuration is sometimes referred to as an open supermembrane. It was shown in [36] that the parallel configuration cannot preserve supercharges compatible with the $N = 1$ supersymmetry on the boundary hyperplanes. This is readily understood once one notices that a such a parallel membrane would correspond to a BPS membrane in ten-dimensional heterotic string theory. However, no such membrane exists and, therefore, parallel membranes are not of interest in this paper. Henceforth, we only consider open supermembrane configurations.

We now show that these configurations do preserve supercharges compatible with the $N = 1$ supersymmetry on the orbifold planes.

We have seen in (2.12) that, in order for supersymmetry to be preserved, the global supersymmetry parameter $\hat{\varepsilon}$ of the membrane worldvolume theory must satisfy $\hat{P}_-\hat{\varepsilon} = 0$, where \hat{P}_- is given in (2.8). An open submembrane is oriented perpendicular to the ten-dimensional hyperplanes. Therefore, we can choose the fields such that

$$\begin{aligned} \hat{X}^{\hat{0}} &= \hat{\sigma}^{\hat{0}}, & \hat{X}^{\hat{1}} &= \hat{\sigma}^{\hat{1}}, & \hat{X}^{\hat{11}} &= \hat{\sigma}^{\hat{2}}, \\ \hat{X}^{\hat{m}} &= 0, & \hat{m} &= 2, 3, \dots, 9 & \hat{\Theta} &= 0, \end{aligned} \quad (2.35)$$

so that $\hat{P}_-\hat{\varepsilon} = 0$ now becomes

$$\hat{P}_-\hat{\varepsilon} = \frac{1}{2}(1 - \hat{\Gamma}_{\hat{0}}\hat{\Gamma}_{\hat{1}}\hat{\Gamma}_{\hat{11}})\hat{\varepsilon} = 0. \quad (2.36)$$

This is as far as one can go in the bulk space. However, on the orbifold boundary planes, (2.31) can be substituted in (2.36) to give

$$\frac{1}{2}(1 - \hat{\Gamma}_{\hat{0}\hat{1}})\hat{\varepsilon} = 0, \quad \text{at } \hat{x}^{\hat{11}} = 0, \pi\rho. \quad (2.37)$$

This expression implies that the eleven-dimensional Majorana spinor $\hat{\varepsilon}$, when restricted to the $1 + 1$ -dimensional boundary strings, is a non-vanishing Majorana-Weyl spinor, as it should be.² We see, therefore, that this configuration preserves supercharges consistent with the supersymmetry on the \mathbb{Z}_2 hyperplanes. Therefore, we conclude that a configuration in which the supermembrane is oriented parallel to the orbifold hyperplanes breaks all supersymmetries. On the other hand, the configuration for the open supermembrane is such that the hyperplane and membrane supersymmetries are compatible. Below, we will analyze the exact role of chiral projection (2.37).

Now assume there is a bulk space five-brane oriented parallel to the orbifold planes, and that the open supermembrane stretches between one orbifold plane and the five-brane. The previous discussion continues to hold on the two-dimensional boundary string, which we denote by $\partial\Sigma_9$, contained in the orbifold plane. What happens at the other two-dimensional boundary string, $\partial\Sigma_5$, embedded in \underline{M}_6 ? Clearly (2.36) must remain true on $\partial\Sigma_5$. It is not hard to show that this expression is compatible with the five-brane spinor chirality constraint (2.33) and, hence, preserves supercharges compatible with the supersymmetry

²When we switch to Euclidean space later in this paper, we must regard $\hat{\varepsilon}$ as an eleven-dimensional Dirac spinor and ε as a ten-dimensional Weyl spinor, since in these dimensions one cannot impose the Majorana condition.

on the orbifold fixed plane. Below, we will analyze the exact role of chiral projection (2.36) on the supercharges of $\partial\Sigma_5$.

We conclude that the configuration consisting of an open supermembrane with two-dimensional boundary strings on an orbifold plane and a parallel five-brane respectively, preserves supercharges consistent with the supersymmetry on the Hořava-Witten fixed hyperplanes.

3 κ -Invariant Action for Open Membranes:

We have shown that for a supermembrane to preserve supersymmetries consistent with the boundary fixed planes and with a parallel oriented five-brane, the membrane must be open. That is, it must be stretched between the two \mathbb{Z}_2 hyperplanes or, as we will be concerned with in this paper, between one of the \mathbb{Z}_2 hyperplanes and the bulk space five-brane. In this section, we want to find the action associated with such a membrane. Action (2.3) is a good starting point. However, it is not obvious that it will correspond to the desired configuration, even in the bulk space. For this to be the case, one needs to ask whether this action respects the \mathbb{Z}_2 symmetry of Hořava-Witten theory. The answer was provided in [36], where it was concluded that, for an appropriate extension of the \mathbb{Z}_2 symmetry to the worldvolume coordinates and similar constraints for the worldvolume metric, the open supermembrane equations of motion are indeed \mathbb{Z}_2 covariant if the supergravity background is \mathbb{Z}_2 invariant. Therefore, we can retain action (2.3). Does it suffice, however, to completely describe the open membrane configuration? Note that the intersection, which we denote by $\partial\Sigma_9$, of one end of the open membrane with the orbifold fixed plane is a two-dimensional string embedded in the ten-dimensional boundary plane M_{10} . We denote by σ^i , $i = 1, 2$, the worldsheet coordinates of this string. Intuitively, one expects extra fields, which we generically denote by $\phi(\sigma)$, to appear on this boundary string in addition to the bulk fields $\hat{\mathbb{Z}}^{\mathbb{M}}(\hat{\sigma})$. These would naturally couple to the pullback onto the boundary string of the background E_8 super-gauge fields $\mathbb{A}_{\mathbb{M}}$. As we will see in this section, new supermembrane fields are indeed required and form a chiral Wess-Zumino-Witten multiplet of the E_8 gauge group.³ Furthermore, the intersection, which we denote by $\partial\Sigma_5$, of the other end of the open supermembrane with the bulk space five-brane is also a 1 + 1-dimensional string. However, this string is embedded in the six-dimensional five-brane worldvolume \underline{M}_6 . As we will see below, unlike the intersection string on the orbifold plane, it is not necessary to have extra fields on the five-brane intersection string in addition to the bulk fields $\hat{\mathbb{Z}}^{\mathbb{M}}(\hat{\sigma})$. These bulk

³This section follows closely the original proof in [35].

fields suffice, through their derivatives along the worldsheet directions, to couple to the pullback onto the string worldsheet of the five-brane super-two-form fields $\underline{\mathbb{D}}_{\text{RS}}$.

Let us first consider the intersection of the open membrane with the boundary hyperplane. As discussed previously, the supergravity theory of the background fields exhibits both gauge and gravitational anomalies that can only be canceled by modifying the Bianchi identity as in (2.34). Integrating (2.34) along the $\hat{x}^{\hat{1}1}$ direction in the neighborhood of the orbifold plane, and promoting the result to superspace, we find that

$$\hat{\mathbb{G}}_{\text{MNPQ}}|_{M_{10}} = -\frac{1}{8\pi T_M}(\text{tr}\mathbb{F} \wedge \mathbb{F})_{\text{MNPQ}}, \quad (3.1)$$

where \mathbb{F} is the super-field-strength of the fields \mathbb{A} . Note that we have dropped the curvature term proportional to $\text{tr}R \wedge R$. The reason for this is that this term is associated, for anomaly cancellation, with higher dimensional terms in the eleven-dimensional supergravity action. However, the brane actions used here couple only to the background fields whose dynamics are given by the usual, low dimensional supergravity theory. Hence, the $\text{tr}R \wedge R$ terms are of higher order from this point of view. The reason for expressing the integrated Bianchi identity in superspace is to make it compatible with the bulk part of supermembrane action (2.3), which is written in terms of the pullbacks of superfields $\hat{\mathbb{E}}_{\hat{\mathbb{M}}}^{\hat{\mathbb{A}}}$ and $\hat{\mathbb{C}}_{\hat{\mathbb{C}}\hat{\mathbb{B}}\hat{\mathbb{A}}}$ onto the worldvolume. Recalling that, locally, $\hat{\mathbb{G}} = d\hat{\mathbb{C}}$, it follows from (3.1) that on the orbifold plane

$$\hat{\mathbb{C}}_{\text{MNP}}|_{M_{10}} = -\frac{1}{8\pi T_M}\Omega_{\text{MNP}}(\mathbb{A}), \quad (3.2)$$

where

$$\Omega_{\text{MNP}}(\mathbb{A}) = 3! \left(\text{tr}(\mathbb{A} \wedge d\mathbb{A}) + \frac{2}{3} \text{tr}(\mathbb{A} \wedge \mathbb{A} \wedge \mathbb{A}) \right)_{\text{MNP}} \quad (3.3)$$

is the Chern-Simons three-form of the super-one-form \mathbb{A} .

Note that each \mathbb{A} is a super-gauge-potential and, as such, transforms under super-gauge transformations as

$$\delta_{\mathbb{L}}\mathbb{A}_{\mathbb{M}}^a = \partial_{\mathbb{M}}\mathbb{L}^a + f^{abc}\mathbb{A}_{\mathbb{M}}^b\mathbb{L}^c, \quad (3.4)$$

with $a, b, c = 1, \dots, 248$. If we define the pullback of \mathbb{A} as

$$\mathbb{A}_i \equiv \partial_i Z^{\mathbb{M}} \mathbb{A}_{\mathbb{M}}, \quad (3.5)$$

the gauge transformation in superspace (3.4) induces a gauge transformation on the string worldsheet, which acts on the pullback of \mathbb{A} as

$$\delta_{\mathbb{L}}\mathbb{A}_i^a = (D_i\mathbb{L})^a = \partial_i\mathbb{L}^a + f^{abc}\mathbb{A}_i^b\mathbb{L}^c, \quad (3.6)$$

where $\mathbb{L} = \mathbb{L}(\mathbb{Z}^{\mathbb{M}}(\sigma))$. It follows from (3.2), (3.3) and (3.4) that, on the boundary fixed plane,

$$\delta_{\mathbb{L}} \hat{\mathbb{C}}_{\text{MNP}}|_{M_{10}} = -\frac{3}{4\pi T_M} \left[\delta_{\mathbb{L}} \left(\text{tr}(\mathbb{A} \wedge d\mathbb{A}) + \frac{2}{3} \text{tr}(\mathbb{A} \wedge \mathbb{A} \wedge \mathbb{A}) \right)_{\text{MNP}} \right] = -\frac{3}{4\pi T_M} \text{tr}(\partial_{[\text{M}} \mathbb{L} \partial_{\text{N}} \mathbb{A}_{\text{P}]}). \quad (3.7)$$

Note that, since the supergauge fields \mathbb{A} live only on the orbifold plane,

$$\delta_{\mathbb{L}} \hat{\mathbb{C}}_{\text{MNP}} = 0 \quad (3.8)$$

everywhere else on the open supermembrane, including its intersection with the bulk space five-brane. Now consider the variation of the supermembrane action (2.3) under a supergauge transformation. Clearly, a non-zero variation arises from the second term in (2.3)

$$\begin{aligned} \delta_{\mathbb{L}} S_{SM} &= \frac{T_M}{6} \int_{\Sigma} d^3 \hat{\sigma} \, \varepsilon^{ij\hat{k}} \partial_i \hat{\mathbb{Z}}^{\hat{\mathbb{M}}} \partial_j \hat{\mathbb{Z}}^{\hat{\mathbb{N}}} \partial_{\hat{k}} \hat{\mathbb{Z}}^{\hat{\mathbb{P}}} \delta_{\mathbb{L}} \hat{\mathbb{C}}_{\hat{\mathbb{P}}\hat{\mathbb{N}}\hat{\mathbb{M}}} \\ &= \frac{1}{8\pi} \int_{\partial\Sigma_9} d^2 \sigma \, \varepsilon^{ij} \partial_i \mathbb{Z}^{\mathbb{M}} \partial_j \mathbb{Z}^{\mathbb{N}} \text{tr}(\mathbb{L} \partial_{\mathbb{N}} \mathbb{A}_{\mathbb{M}}), \end{aligned} \quad (3.9)$$

where we have integrated by parts. Therefore, action (2.3) is not invariant under gauge transformations. This symmetry is violated precisely at the boundary plane. It follows that to restore gauge invariance, one must add an appropriate boundary term to the supermembrane action.

Before doing that, however, let us consider the transformation of the action S_{SM} under a κ -transformation, taking into account the boundary expression (3.2). Note that the κ -transformation acts on the super-three-form $\hat{\mathbb{C}}$ as

$$\delta_{\hat{\kappa}} \hat{\mathbb{C}} = \mathcal{L}_{\hat{\kappa}} \hat{\mathbb{C}} = i_{\hat{\kappa}} d\hat{\mathbb{C}} + (di_{\hat{\kappa}}) \hat{\mathbb{C}}, \quad (3.10)$$

where $\mathcal{L}_{\hat{\kappa}}$ is the Lie derivative in the $\hat{\kappa}$ -direction and the operator $i_{\hat{\kappa}}$ is defined, for any super- l -form $\hat{\mathbb{O}}$, as

$$\begin{aligned} i_{\hat{\kappa}} \hat{\mathbb{O}} &= \frac{1}{l!} \hat{\mathbb{O}}_{\hat{\mathbb{M}}_1 \dots \hat{\mathbb{M}}_l} i_{\hat{\kappa}} (d\hat{\mathbb{Z}}^{\hat{\mathbb{M}}_1} \wedge \dots \wedge d\hat{\mathbb{Z}}^{\hat{\mathbb{M}}_l}) \\ &= \frac{1}{(l-1)!} \hat{\mathbb{O}}_{\hat{\mathbb{M}}_1 \dots \hat{\mathbb{M}}_{l-1} \hat{\mu}} (\hat{P}_+ \hat{\kappa}^{\hat{\mu}}) (d\hat{\mathbb{Z}}^{\hat{\mathbb{M}}_{l-1}} \wedge \dots \wedge d\hat{\mathbb{Z}}^{\hat{\mathbb{M}}_1}). \end{aligned} \quad (3.11)$$

Importantly, we use the positive projection \hat{P}_+ of $\hat{\kappa}$, as defined in (2.7), in order to remain consistent with the previous choices of supersymmetry orientation. Varying action (2.3) under (3.10), and under the full κ -variations of $\hat{\mathbb{Z}}$, we observe that κ -symmetry is also violated at the boundaries

$$\delta_{\hat{\kappa}} S_{SM} = -\frac{1}{6} T_M \int_{\partial\Sigma} d^2 \sigma \, \varepsilon^{ij} \partial_i \mathbb{Z}^{\mathbb{M}} \partial_j \mathbb{Z}^{\mathbb{N}} \mathbb{C}_{\text{NM}\hat{\mu}} \hat{P}_+ \hat{\kappa}^{\hat{\mu}} \quad (3.12)$$

where $\partial\Sigma = \partial\Sigma_9 + \partial\Sigma_5$ is the sum over the two strings on the boundary of the open supermembrane. Using (3.2), this can be written as

$$\begin{aligned}\delta_{\hat{\kappa}} S_{SM} &= \frac{1}{48\pi} \int_{\partial\Sigma_9} d^2\sigma \varepsilon^{ij} \partial_i \mathbb{Z}^{\mathbb{M}} \partial_j \mathbb{Z}^{\mathbb{N}} \Omega_{\mathbb{NM}\hat{\mu}}(\mathbb{A}) \hat{P}_+ \hat{\kappa}^{\hat{\mu}} \\ &\quad - \frac{1}{6} T_M \int_{\partial\Sigma_5} d^2\sigma \varepsilon^{ij} \partial_i \mathbb{Z}^{\mathbb{M}} \partial_j \mathbb{Z}^{\mathbb{N}} \mathbb{C}_{\mathbb{NM}\hat{\mu}} \hat{P}_+ \hat{\kappa}^{\hat{\mu}}\end{aligned}\quad (3.13)$$

In deriving this expression, we have used the eleven-dimensional supergravity constraints. It proves convenient to consider, instead of this κ -transformation, the modified κ -transformation

$$\Delta_{\hat{\kappa}} = \delta_{\hat{\kappa}} - \delta_{\mathbb{L}_{\hat{\kappa}}}, \quad (3.14)$$

where $\delta_{\mathbb{L}_{\hat{\kappa}}}$ is a super-gauge transformation with the special gauge parameter

$$\mathbb{L}_{\hat{\kappa}} = i_{\hat{\kappa}} \mathbb{A} = 2\mathbb{A}_{\hat{\mu}} \hat{P}_+ \hat{\kappa}^{\hat{\mu}}. \quad (3.15)$$

Note that, although this modifies the transformation of the three-form $\hat{\mathbb{C}}_{\mathbb{MNP}}$ on $\partial\Sigma_9$, it follows from (3.8) that this modified transformation is identical to the original κ -transformation everywhere else, including $\partial\Sigma_5$. Under this transformation, the supermembrane action behaves as

$$\begin{aligned}\Delta_{\hat{\kappa}} S_{SM} &= \frac{1}{8\pi} \int_{\partial\Sigma_9} d^2\sigma \varepsilon^{ij} \partial_i \mathbb{Z}^{\mathbb{M}} \partial_j \mathbb{Z}^{\mathbb{N}} \mathbb{F}_{\mathbb{N}\hat{\mu}} \hat{P}_+ \hat{\kappa}^{\hat{\mu}} \mathbb{A}_{\mathbb{M}} \\ &\quad - \frac{1}{6} T_M \int_{\partial\Sigma_5} d^2\sigma \varepsilon^{ij} \partial_i \mathbb{Z}^{\mathbb{M}} \partial_j \mathbb{Z}^{\mathbb{N}} \mathbb{C}_{\mathbb{NM}\hat{\mu}} \hat{P}_+ \hat{\kappa}^{\hat{\mu}}\end{aligned}\quad (3.16)$$

It is important to note that modified κ -invariance of S_{SM} has two independent obstructions, one on the orbifold boundary string $\partial\Sigma_9$ and one on the five-brane string $\partial\Sigma_5$. Both of these, along with the obstruction to gauge invariance on $\partial\Sigma_9$ specified in (3.9), must somehow be canceled if the theory is to be consistent. Before doing that, it is also useful to record that the pullback of the boundary background field \mathbb{A} transforms as

$$\Delta_{\hat{\kappa}} \mathbb{A}_i = 2\partial_i \mathbb{Z}^{\mathbb{M}} \mathbb{F}_{\mathbb{M}\hat{\mu}} \hat{P}_+ \hat{\kappa}^{\hat{\mu}} \quad (3.17)$$

under this modified κ -transformation, where we have used the fact that

$$\delta_{\hat{\kappa}} \mathbb{A} = \mathcal{L}_{\hat{\kappa}} \mathbb{A} \quad (3.18)$$

is the κ -transformation of \mathbb{A} , just as in (3.10).

We now turn to the question of canceling both the gauge and modified κ -transformations on both $\partial\Sigma_9$ and $\partial\Sigma_5$. We begin with the intersection string $\partial\Sigma_9$ of the orbifold plane and the membrane. It was shown in [35] that the gauge and modified κ -anomalies on

$\partial\Sigma_9$, given in (3.9) and the first term in (3.16), can be canceled if the supermembrane action is augmented to include a chiral level one Wess-Zumino-Witten model on the orbifold boundary string of the membrane. The fields thus introduced will couple to the pullback of the background field \mathbb{A} at that boundary.

On $\partial\Sigma_9$, the new fields can be written as

$$g(\sigma) = e^{\phi^a(\sigma)T^a}, \quad (3.19)$$

where T^a , $a = 1, \dots, 248$ are the generators of E_8 and $\phi^a(\sigma)$ are scalar fields that transform in the adjoint representation, and parametrize the group manifold, of E_8 . Note that g is a field living on the worldsheet of the orbifold boundary string. The left-invariant Maurer-Cartan one-forms $\omega_i(\sigma)$ are defined by

$$\omega_i = g^{-1}\partial_i g. \quad (3.20)$$

The variation of $g(\sigma)$ under gauge and modified κ -transformations can be chosen to be

$$\delta_{\mathbb{L}}g = g\mathbb{L}, \quad \Delta_{\hat{\kappa}}g = 0, \quad (3.21)$$

where $\mathbb{L} = \mathbb{L}(\mathbb{Z}(\sigma))$. The coupling of this model to the external gauge fields is accomplished by replacing the left-invariant Maurer-Cartan one-form $\omega_i = g^{-1}\partial_i g$ by the “gauged” version

$$g^{-1}D_i g = \omega_i - \partial_i \mathbb{Z}^{\mathbb{M}} \mathbb{A}_{\mathbb{M}}, \quad (3.22)$$

where D_i is the covariant derivative for the right-action of the gauge group.

An action that is gauge- and κ -invariant on the membrane bulk space worldvolume and on $\partial\Sigma_9$, but not yet on the five-brane boundary string, can be obtained by adding to the bulk action S_{SM} given in (2.3) the Wess-Zumino-Witten action

$$\begin{aligned} S_{WZW} &= \frac{1}{8\pi} \int_{\partial\Sigma_{10}} d^2\sigma \operatorname{tr} \left[\frac{1}{2} \sqrt{-g} g^{ij} (\omega_i - \partial_i \mathbb{Z}^{\mathbb{M}} \mathbb{A}_{\mathbb{M}}) \cdot (\omega_j - \partial_j \mathbb{Z}^{\mathbb{N}} \mathbb{A}_{\mathbb{N}}) + \varepsilon^{ij} \partial_j \mathbb{Z}^{\mathbb{M}} \omega_i \mathbb{A}_{\mathbb{M}} \right] \\ &\quad - \frac{1}{24\pi} \int_{\mathcal{B}} d^3\hat{\sigma} \varepsilon^{ijk} \Omega_{\hat{k}\hat{j}\hat{i}}(\hat{\omega}), \end{aligned} \quad (3.23)$$

where

$$\Omega_{\hat{k}\hat{j}\hat{i}}(\hat{\omega}) = \operatorname{tr}(\hat{\omega} \wedge \hat{\omega} \wedge \hat{\omega})_{\hat{k}\hat{j}\hat{i}} \quad (3.24)$$

and we use $\hat{\varepsilon}_{012} = +1$. The first term in (3.23) describes the kinetic energy for the scalar fields $\phi^a(\sigma)$ and their interactions with the pullback of the super-gauge potential \mathbb{A} . The second term is the integral, over the three-ball \mathcal{B} with boundary $\partial\Sigma_9$, of the Wess-Zumino-Witten three-form, constructed in (3.24) from a one-form $\hat{\omega} = \hat{g}^{-1}d\hat{g}$, where $\hat{g} : \mathcal{B} \rightarrow E_8$. The map \hat{g} must satisfy

$$\hat{g} |_{\partial\Sigma_9} = g, \quad (3.25)$$

but is otherwise unspecified. That such a \hat{g} exists was shown in [38]. Note that we have implicitly assumed that

$$\partial\Sigma_9 = \mathbb{CP}^1 = S^2, \quad (3.26)$$

as we will do later in this paper when computing the superpotential. It is straightforward to demonstrate that the variation of S_{WZW} under both gauge and local modified κ -transformations, $\delta_{\mathbb{L}}$ and $\Delta_{\hat{\kappa}}$ respectively, exactly cancels the variations of the bulk action S_{SM} given in (3.9) and the first term in (3.16), provided we choose the parameter $\hat{\kappa}$ on $\partial\Sigma_9$ to obey

$$P_- \hat{\kappa} = 0, \quad (3.27)$$

where the projection operators P_{\pm} are defined as

$$P_{\pm} \equiv \frac{1}{2} \left(1 \pm \frac{1}{2\sqrt{-\det g_{ij}}} \varepsilon^{ij} \Pi_i^A \Pi_j^B \Gamma_{AB} \right) \quad (3.28)$$

Note that this is consistent with (2.37). On the orbifold boundary string we can denote $\hat{\kappa}$ by κ . In proving this cancellation, it is necessary to use the super-Yang-Mills constraints on the boundary hyperplane. We conclude that by adding S_{WZW} to the action, we have completely canceled the gauge anomaly (3.9) and the $\partial\Sigma_9$ term in the modified κ -anomaly (3.16). We now turn to the question of canceling the remaining obstruction to modified κ -invariance, namely, the $\partial\Sigma_5$ term in (3.16).

Before doing this, it is necessary to discuss the embedding coordinates of the boundary string on the five-brane. First note that, when restricted to the intersection between the membrane and the five-brane,

$$\mathbb{Z}^{\mathbb{M}}(\sigma) = \underline{\mathbb{Y}}^{\mathbb{M}}(\sigma). \quad (3.29)$$

Furthermore, it follows from the static gauge choice (2.32) of the five-brane that all supercoordinates $\underline{\mathbb{Y}}^{\mathbb{M}}(\sigma)$ vanish except for those given in terms of the intrinsic worldvolume supercoordinates as

$$\underline{\mathbb{Y}}^{\mathbb{R}}(\sigma) = (\xi^r(\sigma), \Theta^{\mu}(\sigma)), \quad (3.30)$$

where Θ^{μ} is a 16-component spinor of (2,0)-supersymmetry in six-dimensions. In terms of the supercoordinates in (3.30), the five-brane tensor supermultiplet (2.26) can be expressed as a super-two-form $\underline{\mathbb{D}}_{\underline{\mathbb{R}\mathbb{S}}}$ satisfying

$$\underline{\mathbb{D}}_{\underline{rs}} |_{\Theta^{\mu}=0} = \underline{D}_{\underline{rs}} \quad (3.31)$$

and another constraint that we will specify below. It is useful to note that the second term in (3.16) can now be written as

$$-\frac{1}{6} T_M \int_{\partial\Sigma_5} d^2\sigma \varepsilon^{ij} \partial_i \underline{\mathbb{Y}}^{\mathbb{R}} \partial_j \underline{\mathbb{Y}}^{\mathbb{S}} \underline{\mathbb{C}}_{\underline{\mathbb{S}\mathbb{R}\mu}} P_+ \kappa^{\mu}, \quad (3.32)$$

where we have used the fact that the string can only move within the five-brane worldvolume and, therefore, $\hat{\underline{k}}$ must be projected as in (2.33).

It was shown in [39] that the $\partial\Sigma_5$ term in (3.16), that is, the obstruction to κ -invariance on the intersection string of the membrane and five-brane, can be canceled if one adds to the action $S_{SM} + S_{WZW}$ another term, supported only on $\partial\Sigma_5$, given by

$$S_5 = \frac{1}{6} T_M \int_{\partial\Sigma_5} d^2\sigma \varepsilon^{ij} \underline{\mathbb{D}}_{ij} \quad (3.33)$$

where $i, j = 0, 1$ and

$$\underline{\mathbb{D}}_{ij} = \text{del}_i \underline{\mathbb{Y}}^{\mathbb{R}} \partial_j \underline{\mathbb{Y}}^{\mathbb{S}} \underline{\mathbb{D}}_{\underline{\text{SR}}} \quad (3.34)$$

is the pullback of the five-brane worldvolume super-two-form $\underline{\mathbb{D}}_{\underline{\text{RS}}}$ onto the boundary string $\partial\Sigma_5$. It is important to note that, unlike the case of the Wess-Zumino-Witten action (3.23), it is not necessary to introduce any new dynamical degrees of freedom on $\partial\Sigma_5$ in order to couple this string to the background five-brane worldvolume two-form superfield. Clearly, this piece of the action does not involve the E_8 supergauge fields of the orbifold boundary in any way and, hence, the gauge anomaly continues to vanish. However, $\underline{\mathbb{D}}_{\underline{\text{RS}}}$ does have a non-trivial transformation under modified κ -transformations. This is given by

$$\Delta_\kappa \underline{\mathbb{D}} = \delta_\kappa \underline{\mathbb{D}} = i_\kappa d\underline{\mathbb{D}} + (di_\kappa) \underline{\mathbb{D}} \quad (3.35)$$

where the action of i_κ on any super-l-form is defined in (3.11). Varying S_5 under the modified κ -transformation, we find that

$$\Delta_{\hat{\kappa}} S_5 = \frac{1}{6} T_M \int_{\partial\Sigma_5} d^2\sigma \varepsilon^{ij} \partial_i \underline{\mathbb{Y}}^{\mathbb{R}} \partial_j \underline{\mathbb{Y}}^{\mathbb{S}} (d\underline{\mathbb{D}})_{\underline{\text{SR}}\underline{\mu}} P_+ \kappa^\mu. \quad (3.36)$$

Using (3.32), we can add this variation to the $\partial\Sigma_5$ term in (3.16) giving

$$\Delta_{\hat{\kappa}} S_{SM} |_{\partial\Sigma_5} + \Delta_{\hat{\kappa}} S_5 = \frac{1}{6} T_M \int_{\partial\Sigma_5} d^2\sigma \varepsilon^{ij} \partial_i \underline{\mathbb{Y}}^{\mathbb{R}} \partial_j \underline{\mathbb{Y}}^{\mathbb{S}} \underline{\mathbb{H}}_{\underline{\text{SR}}\underline{\mu}} P_+ \kappa^\mu, \quad (3.37)$$

where

$$\underline{\mathbb{H}} = d\underline{\mathbb{D}} - \underline{\mathbb{C}}. \quad (3.38)$$

This variation will vanish and modified κ -invariance will be restored if and only if we impose the constraint

$$\underline{\mathbb{H}}_{\underline{\text{SR}}\underline{\mu}} = 0. \quad (3.39)$$

This constraint reduces the large reducible multiplet in $\underline{\mathbb{D}}_{\underline{\text{RS}}}$ to the irreducible tensor supermultiplet specified in (2.26).

Thus far, we have demonstrated that the combined action for the open supermembrane with boundary strings

$$S_{OM} = S_{SM} + S_{WZW} + S_5 \quad (3.40)$$

is invariant under both E_8 gauge transformations on $\partial\Sigma_9$ and modified κ -transformations everywhere. We have, however, yet to check that this action is invariant under the Abelian transformations

$$\delta_\Lambda \underline{\mathbb{D}} = d\underline{\Lambda} \quad (3.41)$$

of the super two-form on $\partial\Sigma_5$. As in the case of E_8 supergauge transformations, there are two potential sources of δ_Λ anomalies. First, integrate the modified Bianchi identity (2.34) along the \hat{x}^{11} direction in the neighborhood of the five-brane. We find that

$$\hat{\mathbb{G}}_{MNPQ} |_{\underline{M}_6} = 0 \quad (3.42)$$

Exactly as for the $\text{tr} R \wedge R$ term at the orbifold boundary M_{10} , we have dropped the contribution from the five-brane source J_5 on the right hand side of (2.34). Again, the reason for doing this is that this term is associated, for anomaly cancellation, with higher dimensional terms in the eleven-dimensional supergravity action. However, the five-brane action used here couples only to background fields associated with the usual, low dimension terms in this supergravity theory. Hence, the J_5 terms are higher order from this point of view. It follows from (3.42) that

$$\hat{\mathbb{C}}_{MNP} |_{M_5} = (d\tilde{\mathbb{D}})_{MNP} \quad (3.43)$$

where $\tilde{\mathbb{D}}$ is some super-two-form on \underline{M}_6 unrelated to $\underline{\mathbb{D}}$. Note that this expression is consistent with the constraint equation (3.39). It follows that

$$\delta_\Lambda \mathbb{C}_{MNP} |_{\underline{M}_6} = 0 \quad (3.44)$$

and, hence, S_{SM} is invariant. Second, note that since the variation of its integrand is a total divergence, S_5 is invariant under Abelian transformation (3.41). Combining these results, we conclude that

$$\delta_\Lambda S_{OM} = 0. \quad (3.45)$$

4 Low-Energy Limit and the Heterotic Superstring:

In this paper, we are interested in obtaining an effective four-dimensional theory with $N = 1$ supersymmetry. In particular, we want to compute non-perturbative corrections to the superpotential of the theory. These corrections arise from the non-perturbative interaction

between the background and the open supermembrane embedded in it. The total action of this theory is

$$S_{Total} = S_{HW} + S_{OM} = (S_{SG} + S_{YM}) + (S_{SM} + S_{WZW} + S_5) \quad (4.1)$$

where S_{SG} , S_{YM} can be found in [25] and S_{SM} , S_{WZW} and S_5 are given in (2.3), (3.23) and (3.33) respectively. In addition to compactifying on S^1/\mathbb{Z}_2 , which takes eleven-dimensional supergravity to the Hořava-Witten theory, there must be a second dimensional reduction on a real six-dimensional manifold. This space, which reduces the theory from ten- to four-dimensions on each orbifold boundary plane, and from eleven- to five-dimensions in the bulk space, is taken to be a Calabi-Yau threefold, denoted CY_3 . A Calabi-Yau space is chosen since such a configuration will preserve $N = 1$ supersymmetry in four-dimensions. That is, we now consider M -theory, open supermembranes and five-branes on the geometrical background

$$M_{11} = R_4 \times CY_3 \times S^1/\mathbb{Z}_2 \quad (4.2)$$

where R_4 is four-dimensional, flat space.

It is essential that this theory be Lorentz invariant in four-dimensions. Consider a five-brane located in the bulk space and oriented parallel to the orbifold fixed planes. It is clear that to maintain Lorentz invariance, the manifold of the five-brane must be of the form

$$\underline{M}_6 = R_4 \times \mathcal{C} \quad (4.3)$$

where \mathcal{C} is a real two-dimensional surface with the property that

$$\mathcal{C} \subset CY_3 \quad (4.4)$$

It was shown in [9] that, in order to preserve $N = 1$ four-dimensional supersymmetry on R_4 , \mathcal{C} must be a holomorphic curve in CY_3 . Now consider an open supermembrane stretched between one orbifold plane and the bulk space five-brane. Any such membrane must have an embedding geometry given by

$$\Sigma = \tilde{\mathcal{C}} \times I \quad (4.5)$$

where $\tilde{\mathcal{C}}$ is a real, two-dimensional surface and $I \subset S^1/\mathbb{Z}_2$ is the interval in the orbifold direction between the orbifold plane and the five-brane. Clearly, the requirement of four-dimensional Lorentz invariance implies that

$$\tilde{\mathcal{C}} \subset CY_3 \quad (4.6)$$

Since CY_3 is purely space-like, it follows that we must, henceforth, use the Euclidean version of supermembrane theory.⁴ It was shown in [25] that, in order to preserve $N = 1$ supersymmetry in four-dimensions, it is necessary to choose $\tilde{\mathcal{C}}$ to be a holomorphic curve in CY_3 . Clearly, since Σ has a boundary in \underline{M}_6 we must have

$$\Sigma = \mathcal{C} \times I \quad (4.7)$$

In this section, we take the limit as the radius ρ of S^1 becomes small and explicitly compute the open supermembrane theory in this limit. The result will be the heterotic superstring, coupled to one E_8 gauge background and to a Neveu-Schwarz five-brane, embedded in the ten-dimensional space

$$M_{10} = R_4 \times CY_3, \quad (4.8)$$

and wrapped around a holomorphic curve $\mathcal{C} \subset CY_3$.

We begin by rewriting the action (3.40) for an open supermembrane with boundary strings on one orbifold plane and a bulk space five-brane as

$$\begin{aligned} S_{OM} = & T_M \int_{\Sigma} d^3 \hat{\sigma} (\sqrt{\det \hat{\Pi}_i^{\hat{A}} \hat{\Pi}_j^{\hat{B}} \eta_{\hat{A}\hat{B}}} - \frac{i}{6} \hat{\varepsilon}^{ijk} \hat{\Pi}_i^{\hat{A}} \hat{\Pi}_j^{\hat{B}} \hat{\Pi}_k^{\hat{C}} \hat{\mathbb{C}}_{\hat{C}\hat{B}\hat{A}}) \\ & - \frac{1}{8\pi} \int_{\partial \Sigma_9} d^2 \sigma \operatorname{tr} [\frac{1}{2} \sqrt{g} g^{ij} (\omega_i - \mathbb{A}_i) \cdot (\omega_j - \mathbb{A}_j) + i \varepsilon^{ij} \omega_i \mathbb{A}_j] \\ & + \frac{1}{24\pi} \int_{\mathcal{B}} d^3 \hat{\sigma} i \varepsilon^{\hat{i}\hat{j}\hat{k}} \Omega_{\hat{k}\hat{j}\hat{i}}(\hat{\omega}) - \frac{1}{6} T_M \int_{\partial \Sigma_5} d^2 \sigma i \varepsilon^{ij} \underline{\mathbb{D}}_{ij}. \end{aligned} \quad (4.9)$$

An i appears multiplying the epsilon symbols because we are in Euclidean space. Furthermore, it is important to note that the requirement that we work in Euclidean space changes the sign of each term in (4.9) relative to the Minkowski signature action given by (3.23) and (3.33). The boundary terms describe the gauged chiral Wess-Zumino-Witten model on the orbifold string and the coupling to the super-two-form on the five-brane string. Since they are defined only on the boundary, they are not affected by the compactification on S^1/\mathbb{Z}_2 . As for the bulk action, we identify

$$\hat{X}^{\hat{1}\hat{1}} = \hat{\sigma}^{\hat{2}} \quad (4.10)$$

and for all remaining fields keep only the dependence on $\hat{\sigma}^{\hat{0}}, \hat{\sigma}^{\hat{1}}$. The explicit reduction of the bulk action was carried out in [25], to which we refer the reader. Here, we will simply state the result. We find that the first part of action (4.9) reduces in the small ρ limit to the string action

$$S_S = T_S \frac{Y}{\pi \rho} \int_{\mathcal{C}} d^2 \sigma (\phi \sqrt{\det \Pi_i^A \Pi_j^B \eta_{AB}} - \frac{i}{2} \varepsilon^{ij} \Pi_i^{\mathbb{A}} \Pi_j^{\mathbb{B}} \mathbb{B}_{\mathbb{B}\mathbb{A}}), \quad (4.11)$$

⁴Another reason to Euclideanize the theory is that, in this paper, we will perform the calculation of quantum corrections using the path-integral formalism.

where

$$T_S = T_M \pi \rho \equiv (2\pi\alpha')^{-1} \quad (4.12)$$

is the string tension of mass dimension two, super-two-form $\mathbb{B}_{\mathbb{B}\mathbb{A}}$ and dilaton superfield ϕ are defined below and Y is the location coordinate of the five-brane in the S^1/\mathbb{Z}_2 orbifold interval. This coordinate is chosen so that when $Y \rightarrow 0$, the length of the open membrane shrinks to zero. This important factor arises from the fact that, by assumption, no fields depend on intrinsic coordinate $\hat{\sigma}^2$ and that

$$\int d^3\hat{\sigma} = \int_0^Y d\hat{\sigma}^2 \int d^2\sigma = Y \int d^2\sigma \quad (4.13)$$

Before we can write the total action for the open supermembrane compactified on S^1/\mathbb{Z}_2 , we must discuss the boundary terms in (4.9). In the limit that the radius ρ of S^1 shrinks to zero, the orbifold fixed plane and the five-brane coincide. Generically, the two different boundaries of the supermembrane need not be identified. However, since our supersymmetric embedding Ansatz (4.10) assumes all quantities to be independent of the orbifold coordinate, the two boundary strings coincide as the zero radius limit is taken. This has further implications beyond the fact that, at low energy, we are dealing with a single string. To see this, begin by considering the full orbifold before taking the small ρ limit and before compactifying on CY_3 . Note that, prior to the embedding Ansatz, the membrane boundary on the orbifold fixed plane, $\partial\Sigma_9$, can be any two-dimensional subset of M_{10} . However, Ansatz (4.10) implies that

$$\partial\Sigma_9 \subset \underline{M}_6 \subset M_{10}, \quad (4.14)$$

where \underline{M}_6 is the induced embedding of the five-brane manifold into M_{10} . This constraint limits the bosonic target space coordinates of $\partial\Sigma_9$ to lie in a six-dimensional submanifold of M_{10} and will have important implications that will be discussed later in this paper. Furthermore, the restriction of $\partial\Sigma_9$ to $\underline{M}_6 \subset M_{10}$ implies that the five-brane chiral constraint (2.33) now applies to the supercharges on $\partial\Sigma_9$, in addition to the \mathbb{Z}_2 induced chiral constraint (2.31). This reduces from 16 to 8 the number of preserved supercharges on $\partial\Sigma_9$. The embedding Ansatz, however, prior to taking the small ρ limit has no effect on the coordinates, bosonic or fermionic, of $\partial\Sigma_5$.

Now take the limit that $\rho \rightarrow 0$. In this limit, there is no change on $\partial\Sigma_9$. However, in the small radius limit, the \mathbb{Z}_2 projection (2.31) applies to supercharges on $\partial\Sigma_5$ in addition to the chiral constraint (2.33), reducing them from 16 to 8. They are given by exactly the same supercharges as on $\partial\Sigma_9$. The small ρ limit does not affect the bosonic coordinates of $\partial\Sigma_5$ which, by definition, will satisfy $\partial\Sigma_5 \subset \underline{M}_6 \subset M_{10}$.

Note that our analysis of the supercharges of both $\partial\Sigma_9$ and $\partial\Sigma_5$, in the small ρ limit, remains incomplete. As discussed previously, prior to taking ρ small, the supercharges on $\partial\Sigma_9$ are further restricted by chiral constraint (2.37) and those on $\partial\Sigma_5$ by chiral constraint (2.36). In the $\rho \rightarrow 0$ limit, these constraints become identical. This constraint further reduces the number of supercharges from 8 to 4. We conclude that, as $\rho \rightarrow 0$, the boundary strings coincide so that

$$\mathcal{C} = \partial\Sigma_9 = \partial\Sigma_5 \quad (4.15)$$

and satisfy

$$\mathcal{C} \subset \underline{M}_6 \subset M_{10} \quad (4.16)$$

with four preserved supercharges. These restrictions are important, as we will see below. Putting everything together, we find that the resulting action is

$$\begin{aligned} S_{\mathcal{C}} = & T_S \frac{Y}{\pi\rho} \int_{\mathcal{C}} d^2\sigma (\phi \sqrt{\det \Pi_i^A \Pi_j^B \eta_{AB}} - \frac{i}{2} \varepsilon^{ij} \Pi_i^{\mathbb{A}} \Pi_j^{\mathbb{B}} \mathbb{B}_{\mathbb{A}\mathbb{B}}) \\ & - \frac{1}{8\pi} \int_{\mathcal{C}} d^2\sigma \operatorname{tr} [\frac{1}{2} \sqrt{g} g^{ij} (\omega_i - \mathbb{A}_i) \cdot (\omega_j - \mathbb{A}_j) + i \varepsilon^{ij} \omega_i \mathbb{A}_j] \\ & + \frac{1}{24\pi} \int_B d^3\hat{\sigma} i \hat{\varepsilon}^{\hat{i}\hat{j}\hat{k}} \Omega_{\hat{k}\hat{j}\hat{i}}(\hat{\omega}) - \frac{1}{6} T_S \int_{\mathcal{C}} d^2\sigma i \varepsilon^{ij} \underline{\mathbb{D}}_{ij}, \end{aligned} \quad (4.17)$$

where

$$\Pi_i^{\mathbb{A}} = \partial_i \mathbb{Z}^{\mathbb{M}} \mathbb{E}_{\mathbb{M}}^{\mathbb{A}}. \quad (4.18)$$

and

$$\mathbb{B}_{\mathbb{M}\mathbb{N}} = \hat{\mathbb{C}}_{\mathbb{M}\mathbb{N}\hat{1}\hat{1}}, \quad \phi = \hat{\mathbb{E}}_{\hat{1}\hat{1}}^{\hat{1}\hat{1}}. \quad (4.19)$$

Note that in the last term of (4.17) we have used (4.12) and absorbed a factor of $1/\pi\rho$ into the definition of the superfield $\underline{\mathbb{D}}$ so that it now has mass dimension zero. For ease of notation, we have written (4.17) in terms of the ten-dimensional superembedding coordinates

$$\mathbb{Z}^{\mathbb{M}} = (X^M, \Theta^\mu), \quad (4.20)$$

where spinor Θ satisfies the Weyl chirality constraint

$$\frac{1}{2}(1 - \Gamma_{11})\Theta = 0. \quad (4.21)$$

However, as we have just discussed, the superembedding is to be considered further restricted to

$$\mathbb{Z}^{\mathbb{R}} = \underline{\mathbb{Y}}^{\mathbb{R}} = (\xi^{\underline{r}}, \Theta^{\underline{\mu}}), \quad (4.22)$$

where $\underline{r} = 0, 1, \dots, 5$ and Θ satisfies the additional, gauge-fixing conditions that

$$\frac{1}{2}(1 + i\Gamma_{012345})\Theta = 0, \quad \frac{1}{2}(1 + i\Gamma_{01})\Theta = 0. \quad (4.23)$$

The chiral projections in (4.21) and (4.23) reduce the number of independent components of spinor Θ to four. We note in passing that the dilaton superfield ϕ satisfies

$$\hat{g}_{\hat{1}\hat{1}\hat{1}\hat{1}} = \phi^2. \quad (4.24)$$

This expression will be useful in the next section when discussing low energy moduli fields. We recognize the action (4.17) as that of the heterotic superstring coupled to one E_8 gauge background, a Neveu-Schwarz five-brane and wrapped on a holomorphic curve $\mathcal{C} \subset CY_3$. In this paper, the curve \mathcal{C} is restricted to

$$\mathcal{C} = \mathbb{CP}^1 = S^2. \quad (4.25)$$

This follows from expressions (3.26) and (4.15).

5 Superpotential in 4D Effective Field Theory:

It is essential when constructing superpotentials to have a detailed understanding of all the moduli in five-dimensional heterotic M -theory. Furthermore, we must know explicitly how they combine to form the moduli of the four-dimensional low-energy theory. The compactification of Hořava-Witten theory to heterotic M -theory on a Calabi-Yau threefold with G -flux, but without bulk five-branes, was carried out in [6, 8], and reviewed in [40]. The further compactification of this theory on S^1/\mathbb{Z}_2 , arriving at the $N = 1$ supersymmetric action of the effective four-dimensional theory was presented originally in [8] and, again, was reviewed in [40]. We refer the reader to these papers for all necessary details. Here, we discuss only those relevant moduli not reviewed in [40], namely, the moduli associated with the translation of the bulk-space five-brane. We emphasize that, throughout this paper, we take the bosonic components of all superfields to be of dimension zero, both in five-dimensional heterotic M -theory and in the associated four-dimensional effective theory.

First, consider the compactification from Hořava-Witten theory to heterotic M -theory. This compactification is carried out as follows. Consider the metric

$$ds_{11}^2 = V^{-2/3} g_{\hat{u}\hat{v}} d\hat{y}^{\hat{u}} d\hat{y}^{\hat{v}} + g_{\check{U}\check{V}} d\check{y}^{\check{U}} d\check{y}^{\check{V}}, \quad (5.1)$$

where $\hat{y}^{\hat{u}}$, $\hat{u} = 2, 3, 4, 5, 11$ are the coordinates of the five-dimensional bulk space of heterotic M -theory, $\check{y}^{\check{U}}$, $\check{U} = 0, 1, 6, 7, 8, 9$ are the Calabi-Yau coordinates and $g_{\check{U}\check{V}}$ is the metric on the Calabi-Yau space CY_3 . The factor $V^{-2/3}$ in (5.1) has been chosen so that metric $g_{\hat{u}\hat{v}}$ is the five-dimensional Einstein frame metric. The Calabi-Yau volume modulus $V = V(y^{\hat{u}})$ is defined by

$$V = \frac{1}{v} \int_{CY_3} \sqrt{\check{g}}, \quad (5.2)$$

where \check{g} is the determinant of the Calabi-Yau metric $g_{\check{U}\check{V}}$ and v is a dimensionful parameter necessary to make V dimensionless.

These fields all must be the bosonic components of specific $N = 1$ supermultiplets in five-dimensions. These supermultiplets are easily identified as follows.

1. Supergravity: the bosonic part of this supermultiplet is

$$(g_{\hat{u}\hat{v}}, \mathcal{A}_{\hat{u}}, \dots). \quad (5.3)$$

This accounts for $g_{\hat{u}\hat{v}}$. The origin of the graviphoton component $\mathcal{A}_{\hat{u}}$ was discussed in [8].

2. Universal Hypermultiplet: the bosonic part of this supermultiplet is

$$(V, C_{\hat{u}\hat{v}\hat{w}}, \xi, \dots), \quad (5.4)$$

which accounts for the Calabi-Yau volume modulus V . The remaining zero-modes components were discussed in [8]. Having identified the appropriate $N = 1$, five-dimensional superfields, one can read off the zero-mode fermion spectrum to be precisely those fermions that complete these supermultiplets.

Thus far, we have not said anything about the bulk space five-brane. As discussed in Section 2, after fixing the κ -gauge the worldvolume theory of the five-brane exhibits $(2, 0)$ -supersymmetry. The worldvolume fields of the five-brane form a tensor supermultiplet.

3. Tensor Supermultiplet: The complete supermultiplet is

$$(\underline{D}_{\underline{rs}}, \underline{\hat{Y}}^{\hat{p}}, \chi), \quad \hat{p} = 6, \dots, 9, 11, \quad (5.5)$$

where the field-strength of $\underline{D}_{\underline{rs}}$ is anti-self-dual, there are five scalars $\underline{\hat{Y}}^{\hat{p}}$ and χ are the associated fermions. For a five-brane oriented parallel to the orbifold fixed planes, four of the scalars \underline{Y}^p , $p = 6, \dots, 9$ are moduli in the Calabi-Yau direction and we can ignore them. The fifth scalar $Y^{\hat{11}}$, which we now simply refer to as Y , is the translational mode of the five-brane in the orbifold direction and is of principal interest in this paper. All of these fields are functions of the six worldvolume coordinates $\xi^{\underline{r}}$, $\underline{r} = 0, 1, \dots, 5$.

We now move to the discussion of the compactification of heterotic M -theory in five-dimensions to the effective $N = 1$ supersymmetric theory in four-dimensions. This compactification, without the five-brane, was carried out in detail in [8] and reviewed in [40]. Here, we simply state the relevant four-dimensional zero-modes and their exact relationship to the five-dimensional moduli of heterotic M -theory. The bulk space zero-modes coincide with the \mathbb{Z}_2 -even fields. One finds that the metric is

$$ds_5^2 = R^{-1} g_{uv} dy^u dy^v + R^2 (dy^{\hat{11}})^2, \quad (5.6)$$

where g_{uv} is the four-dimensional metric, $R = R(y^u)$ is the volume modulus of S^1/\mathbb{Z}_2 and y^u , $u = 2, 3, 4, 5$ are the four-dimensional coordinates. The Calabi-Yau volume modulus reduces to

$$V = V(y^u). \quad (5.7)$$

It is conventional to incorporate this field into the complex dilaton S as

$$S = V + i\sqrt{2}\sigma \quad (5.8)$$

where scalar field σ was discussed in [8]. Furthermore, there are an additional $h^{1,1}$ $(1,1)$ -moduli, denoted by T^I , which arise in the context of superpotentials and were defined in detail in [8]. Of importance in this paper is a particular linear combination of these $(1,1)$ -moduli, which we denote by \mathcal{T} . Modulus \mathcal{T} is related to the $(1,1)$ -moduli T^I as follows. Recall that the cohomology group $H^{(1,1)}$ on CY_3 has a basis of harmonic $(1,1)$ -forms ω_I , $I = 1, \dots, h^{1,1}$. These are naturally dual to a basis \mathcal{C}_I , $I = 1, \dots, h^{1,1}$ of curves in $H_{(1,1)}$ where

$$\frac{1}{v_{\mathcal{C}}} \int_{\mathcal{C}_I} \omega_J = \delta_{IJ}. \quad (5.9)$$

We have introduced a parameter $v_{\mathcal{C}}$ of mass dimension minus two to make the integral dimensionless. Parameter $v_{\mathcal{C}}$ can be taken to be the volume of curve \mathcal{C} . Any holomorphic curve can be expressed as a linear combination of the \mathcal{C}_I curves. For example, the curve \mathcal{C} around which our heterotic string is wrapped can be written

$$\mathcal{C} = \sum_{I=1}^{h^{1,1}} c_I \mathcal{C}_I \quad (5.10)$$

for some complex coefficients c_I , $I = 1, \dots, h^{1,1}$. The dual to this expression is the harmonic $(1,1)$ -form

$$\omega_{\mathcal{C}} = \frac{1}{(\sum_{K=1}^{h^{1,1}} c_K^2)} \sum_{I=1}^{h^{1,1}} c_I \omega_I, \quad (5.11)$$

where

$$\frac{1}{v_{\mathcal{C}}} \int_{\mathcal{C}} \omega_{\mathcal{C}} = 1. \quad (5.12)$$

This form can be extended to a basis of $H^{(1,1)}$. Denote the remaining $h^{1,1} - 1$ basis forms by ω'_i , with the property

$$\frac{1}{v_{\mathcal{C}}} \int_{\mathcal{C}} \omega'_i = 0. \quad (5.13)$$

Now, note from the discussion in [25] that

$$RV^{-1/3}\omega = \sum_{I=1}^{h^{1,1}} \text{Re} T^I \omega_I, \quad (5.14)$$

where ω is the Kähler form on CY_3 . Similarly, one can define $\text{Re}\mathcal{T}$ by

$$RV^{-1/3}\omega = \text{Re}\mathcal{T}\omega_{\mathcal{C}} + \sum_{i=1}^{h^{1,1}-1} \beta^i \omega'_i. \quad (5.15)$$

Equating these two expressions and integrating over \mathcal{C} using (5.9), (5.10), (5.12) and (5.13), we find that

$$\text{Re}\mathcal{T} = \sum_{I=1}^{h^{1,1}} c_I \text{Re}T^I. \quad (5.16)$$

Furthermore, from the discussion in [25] we note that

$$B = \sum_{I=1}^{h^{1,1}} \text{Im}T^I \omega_I, \quad (5.17)$$

where $B_{m\bar{n}} = \hat{C}_{m\bar{n}1\hat{1}}$ is the bosonic component of superfield \mathbb{B}_{MN} defined in (4.19). Similarly, one can define $\text{Im}\mathcal{T}$ by

$$B = \text{Im}\mathcal{T}\omega_{\mathcal{C}} + \sum_{i=1}^{h^{1,1}-1} \gamma^i \omega'_i. \quad (5.18)$$

Integrating these two expressions over \mathcal{C} using (5.9), (5.10), (5.12) and (5.13), we find that

$$\text{Im}\mathcal{T} = \sum_{I=1}^{h^{1,1}} c_I \text{Im}T^I. \quad (5.19)$$

Putting equations (5.16) and (5.19) together, we conclude that

$$\mathcal{T} = \sum_{I=1}^{h^{1,1}} c_I T^I. \quad (5.20)$$

The exact form of the four-dimensional $N = 1$ translational supermultiplet of the five-brane has to be carefully discussed at this point. It was shown in [9] that, when a five-brane is compactified to four-dimensions on a holomorphic curve \mathcal{C} of genus g , there are two types of $N = 1$ zero-mode supermultiplets that arise. First, there are g Abelian vector superfields. Since we are concerned with superpotentials in this paper, these superfields are not of interest to us and we will mention them no further. The second type of multiplet that arises is associated with the translational scalar mode, now reduced to

$$Y = Y(y^u). \quad (5.21)$$

In addition, one must consider the four-dimensional modulus associated with the two-form \underline{D}_{rs} . This is found by expanding

$$\underline{D} = 3a\omega_{\mathcal{C}}, \quad (5.22)$$

where $a = a(y^u)$. It was shown in [28], in an entirely different context, that the $N = 1$ translational supermultiplet of the five-brane is a chiral multiplet whose bosonic component is given by

$$\mathbf{Y} = \frac{Y}{\pi\rho} \text{Re}\mathcal{T} + i(a + \frac{Y}{\pi\rho} \text{Im}\mathcal{T}). \quad (5.23)$$

The divisor $\pi\rho$ renders $Y/\pi\rho$ and, hence, \mathbf{Y} dimensionless.

It is then easily seen that these modes form the following four-dimensional, $N = 1$ supermultiplets.

1. Supergravity: the full supermultiplet is

$$(g_{uv}, \psi_u^\alpha), \quad (5.24)$$

where ψ_u^α is the gravitino.

2. Dilaton and T-Moduli Chiral Supermultiplets: the full multiplets are

$$(S, \lambda_S), \quad (T^I, \lambda_T^I), \quad (5.25)$$

where $I = 1, \dots, h^{1,1}$ and λ_S, λ_T^I are the dilatino and T-modulinos, respectively. In particular, the \mathcal{T} modulus is the lowest component of chiral superfield

$$(\mathcal{T}, \lambda_{\mathcal{T}}) \quad (5.26)$$

3. Five-Brane Translation Chiral Supermultiplet: the full multiplet is

$$(\mathbf{Y}, \lambda_{\mathbf{Y}}) \quad (5.27)$$

where $\lambda_{\mathbf{Y}}$ is the associated Weyl fermion. The fermions completing these supermultiplets arise as zero-modes of the fermions of five-dimensional heterotic M -theory. The action for the effective, four-dimensional, $N = 1$ theory has been derived in detail in [8]. Here we simply state the result. The relevant terms for a general discussion of the superpotential are the kinetic terms for the S, T^I and \mathbf{Y} moduli and the bilinear terms of their superpartner fermions. If we collectively denote S, T^I and \mathbf{Y} as $Y^{I'}$, where $I' = 1, \dots, h^{1,1} + 2$, and their fermionic superpartners as $\lambda^{I'}$, then the component Lagrangian is given by

$$\begin{aligned} \mathcal{L}_{4D} = & K_{I' \bar{J}'} \partial_u Y^{I'} \partial^u \bar{Y}^{\bar{J}'} + e^{\kappa_p^2 K} \left(K^{I' \bar{J}'} D_{I'} W \bar{D}_{\bar{J}'} W - 3\kappa_p^2 |W|^2 \right) \\ & + K_{I' \bar{J}'} \lambda^{I'} \not{\partial} \lambda^{\bar{J}'} - e^{\kappa_p^2 K/2} (D_{I'} D_{J'} W) \lambda^{I'} \lambda^{J'} + \text{h.c.} \end{aligned} \quad (5.28)$$

Here κ_p^2 is the four-dimensional Newton's constant,

$$K_{I' \bar{J}'} = \partial_{I'} \partial_{\bar{J}'} K \quad (5.29)$$

are the Kähler metric and Kähler potential respectively, and

$$D_{I'}W = \partial_{I'}W + \kappa_p^2 \frac{\partial K}{\partial Y^{I'}}W \quad (5.30)$$

is the Kähler covariant derivative acting on the superpotential W . The Kähler potential, excluding the five-brane translational mode \mathbf{Y} , was computed in [8]. This result was extended to include \mathbf{Y} in [28]. In terms of the S , T^I and \mathbf{Y} moduli, it is given by

$$\kappa_p^2 K = -\ln(S + \bar{S} - \frac{\tau}{16} \frac{(\mathbf{Y} + \bar{\mathbf{Y}})^2}{\mathcal{T} + \bar{\mathcal{T}}}) - \ln \left(\frac{1}{6} \sum_{I,J,K=1}^{h^{1,1}} d_{IJK} (T + \bar{T})^I (T + \bar{T})^J (T + \bar{T})^K \right), \quad (5.31)$$

where τ is the dimensionless parameter

$$\tau = T_5 v_{\mathcal{C}} (\pi \rho)^2 \kappa_4^2. \quad (5.32)$$

It is useful at this point to relate the low energy fields of the heterotic superstring action derived in Section 4 to the four-dimensional moduli derived here from heterotic M -theory. Specifically, we note from (4.24) that

$$\hat{g}_{\hat{1}\hat{1}\hat{1}\hat{1}}|_{\Theta=0} = \phi^2|_{\Theta=0}, \quad (5.33)$$

and from (5.1) and (5.6) that

$$ds_{11}^2 = \dots + R^2 V^{-2/3} (dy^{\hat{1}\hat{1}})^2. \quad (5.34)$$

Identifying them implies

$$\phi|_{\Theta=0} = R V^{-1/3}. \quad (5.35)$$

We will use this identification in the next section.

Following the approach of [26] and [27], we will calculate the non-perturbative superpotential by computing instanton induced fermion bilinear interactions and then comparing these to the fermion bilinear terms in the low energy effective supergravity action. In this paper, the instanton contribution arises from open supermembranes wrapping on a product of an interval $I \subset S^1/\mathbb{Z}_2$ and a holomorphic curve $\mathcal{C} \subset CY_3$. Specifically, we will calculate this instanton contribution to the two-point function of the fermions $\lambda_{\mathbf{Y}}$ associated with the \mathbf{Y} moduli. The two-point function of four-dimensional space-time fermions $\lambda_{\mathbf{Y}}$ located at positions y_1^u, y_2^u is given by the following path integral expression

$$\langle \lambda_{\mathbf{Y}}(y_1^u) \lambda_{\mathbf{Y}}(y_2^u) \rangle = \int \mathcal{D}\Phi e^{-S_{4D}} \lambda_{\mathbf{Y}}(y_1^u) \lambda_{\mathbf{Y}}(y_2^u) \cdot \int \mathcal{D}\hat{\mathbb{Z}} \mathcal{D}\omega e^{-S_{\Sigma}(\hat{\mathbb{Z}}, \omega; \hat{\mathbb{E}}_{\hat{\mathbb{M}}}^{\hat{\mathbb{A}}}, \hat{\mathbb{C}}_{\hat{\mathbb{M}}\hat{\mathbb{N}}\hat{\mathbb{P}}}, \mathbb{A}_{\mathbb{M}}, \mathbb{D}_{\mathbb{RS}})}, \quad (5.36)$$

where S_Σ is the open supermembrane action given in (4.9). Here Φ denotes all supergravity fields in the $N = 1$ supersymmetric four-dimensional Lagrangian (5.28) and \hat{Z}, ω are the worldvolume fields on the open supermembrane. In addition, the path-integral is performed over all supersymmetry preserving configurations, $(\hat{E}_M^{\hat{A}}, \hat{C}_{\hat{M}\hat{N}\hat{P}}, A_M, \underline{D}_{\underline{RS}})$, of the membrane in the eleven-dimensional Hořava-Witten background with a bulk five-brane, compactified down to four-dimensions on $CY_3 \times S^1/\mathbb{Z}_2$. The integration will restore $N = 1$ four-dimensional supersymmetry. The result of this calculation is then compared to the terms in (5.28) proportional to $(D_Y D_Y W) \lambda_Y \lambda_Y$ and the non-perturbative contribution to W extracted.

6 String Action Expansion:

In this paper, we are interested in the non-perturbative contributions of open supermembrane instantons to the two-point function (5.36) of chiral fermions in the four-dimensional effective field theory. In order to preserve $N = 1$ supersymmetry, the supermembrane must be of the form $\Sigma = \mathcal{C} \times I$, where curve $\mathcal{C} \subset CY_3$ is holomorphic and $I \subset S^1/\mathbb{Z}_2$. As we have shown in previous sections, this is equivalent, in the low energy limit, to considering the non-perturbative contributions of heterotic superstring instantons to the same fermion two-point function in the effective four-dimensional theory. Of course, in this setting, the superstring must wrap completely around a holomorphic curve $\mathcal{C} \subset CY_3$ in order for the theory to be $N = 1$ supersymmetric.

Since we are interested only in non-perturbative corrections to the two-point function $\langle \lambda_Y(y_1^u) \lambda_Y(y_2^u) \rangle$, the perturbative contributions to this function, which arise from the interaction terms in the effective four-dimensional action S_{4D} in (5.36), will not be considered in this paper. Therefore, we keep only the kinetic terms of all four-dimensional dynamic fields in S_{4D} . Furthermore, we can perform the functional integrations over all these fields except λ_Y , obtaining some constant determinant factors which we need not evaluate. Therefore, we can rewrite (5.36) as

$$\begin{aligned} \langle \lambda_Y(y_1^u) \lambda_Y(y_2^u) \rangle &\propto \int \mathcal{D}\lambda_Y e^{-\int d^4y \lambda_Y \not{\partial} \lambda_Y} \lambda_Y(y_1^u) \lambda_Y(y_2^u) \\ &\cdot \int \mathcal{D}Z \mathcal{D}\omega e^{-S_C(Z, \omega; E_M^{\hat{A}}, B_{\hat{M}\hat{N}}, \phi, A_M, \underline{D}_{\underline{RS}})}, \end{aligned} \quad (6.1)$$

where S_C is the heterotic superstring action given in (4.17). As we will see shortly, the functional dependence of S_C on the fields λ_Y comes from the interaction between the superstring fermionic field Θ and the five-brane fermion \mathcal{X} (from which λ_Y is derived in the

compactification). Recall that both of these fermions are Weyl spinors in ten-dimensions.⁵

Clearly, to perform the computation of the two-point function (6.1), we must write the action S_C in terms of its dynamical fields and their interactions with the dimensionally reduced background fields. This means that we must first expand all superfield expressions in terms of component fields. We will then expand the action in small fluctuations around its extrema (solutions to the superstring equations of motion), corresponding to a saddle-point approximation. We will see that because there exists two fermionic zero-modes arising from Θ , their interaction with the five-brane fermion \mathcal{X} will produce a non-vanishing contribution to (6.1). Therefore, when performing the path-integrals over the superstring fields, we must discuss the zero-modes with care. The next step will be to consider the expression for the superstring action and to write it in terms of the complex five-brane translation modulus. Finally, we will perform all remaining path integrals in the saddle-point approximation, obtaining the appropriate determinants.

We start by expanding the ten-dimensional superfields in the action S_C in terms of the component fields.

Expanding in Powers of Θ :

In this section, for ease of notation, we take the superembedding coordinates to be $\mathbb{Z} = (X, \Theta)$ where $(1 - \Gamma_{11})\Theta = 0$ as in (4.21). The required restrictions of X to ξ and Θ to satisfy $(1 + i\Gamma_{012345})\Theta = 0$, as in (4.23), will be carried out in the next section along with further gauge fixing choices.

We begin by rewriting action S_C in (4.17) as

$$S_C = S_S + S_5 + S_{WZW}, \quad (6.2)$$

where

$$\begin{aligned} S_S(\mathbb{Z}; \mathbb{E}_M^A(\mathbb{Z}), \mathbb{B}_{MN}(\mathbb{Z}), \phi(\mathbb{Z})) &= T_S \frac{Y}{\pi\rho} \int_C d^2\sigma (\phi \sqrt{\det \partial_i \mathbb{Z}^M \mathbb{E}_M^A \partial_j \mathbb{Z}^N \mathbb{E}_N^B \eta_{AB}} \\ &\quad - \frac{i}{2} \varepsilon^{ij} \partial_i \mathbb{Z}^M \mathbb{E}_M^A \partial_j \mathbb{Z}^N \mathbb{E}_N^B \mathbb{B}_{BA}) \end{aligned} \quad (6.3)$$

is the supermembrane bulk action dimensionally reduced on $I \subset S^1/\mathbb{Z}_2$,

$$S_5(\mathbb{Z}; \mathbb{D}_{\mathbb{RS}}(\mathbb{Z})) = \frac{i}{6} T_S \int_C d^2\sigma \varepsilon^{ij} \partial_i \mathbb{Z}^R \partial_j \mathbb{Z}^S \mathbb{D}_{\mathbb{SR}} \quad (6.4)$$

is the action of the boundary string where the membrane meets the five-brane and

$$S_{WZW}(\mathbb{Z}, \omega; \mathbb{A}_M(\mathbb{Z})) = -\frac{1}{8\pi} \int_C d^2\sigma \operatorname{tr} \left[\frac{1}{2} \sqrt{g} g^{ij} (\omega_i - \mathbb{A}_i) \cdot (\omega_j - \mathbb{A}_j) + i \varepsilon^{ij} \omega_i \mathbb{A}_j \right]$$

⁵Note that in Euclidean space one does not have Majorana-Weyl spinors in ten-dimensions.

$$+\frac{1}{24\pi}\int_{\mathcal{B}}d^3\sigma i\hat{\varepsilon}^{\hat{i}\hat{j}\hat{k}}\Omega_{\hat{k}\hat{j}\hat{i}}(\hat{\omega}). \quad (6.5)$$

is the gauged Wess-Zumino-Witten action on the other boundary string, where

$$\mathbb{A}_i = \partial_i \mathbb{Z}^{\mathbb{M}} \mathbb{A}_{\mathbb{M}}(\mathbb{Z}). \quad (6.6)$$

Note that this action is a functional of $\mathbb{Z}(\sigma) = (X(\sigma), \Theta(\sigma))$. We now want to expand the superfields in (6.2) in powers of the fermionic coordinate $\Theta(\sigma)$. For the purposes of this paper, we need only keep terms up to second order in Θ . We begin with $S_S + S_5$ given in (6.3) and (6.4). Using an approach similar to [41] and using the results in [42], we find that, to the order in Θ required, the super-zehnbeins are given by

$$\mathbb{E}_{\mathbb{M}}^{\mathbb{A}} = \begin{pmatrix} E_M^{\mathbb{A}} & \frac{1}{4}\omega_M^{CD}(\Gamma_{CD})_{\nu}^{\alpha}\Theta^{\nu} \\ -i\Gamma_{\mu\nu}^{\mathbb{A}}\Theta^{\nu} & \delta_{\mu}^{\alpha} \end{pmatrix}, \quad (6.7)$$

where $E_M^{\mathbb{A}}(X(\sigma))$ are the bosonic zehnbeins and $\omega_M^{CD}(X(\sigma))$ is the ten-dimensional spin connection, defined in terms of derivatives of $E_M^{\mathbb{A}}(X)$. We turn off the gravitino background in this expression for simplicity. We will discuss below its contribution to the two-point function of the fermion related to five-brane translation. The super-two-form fields associated with the membrane are, up to the order in Θ required,

$$\begin{aligned} \mathbb{B}_{MN} &= B_{MN} - \frac{1}{4}\phi\bar{\Theta}\Gamma_{[M}\Gamma^{CD}\Theta\omega_{N]CD}, \\ \mathbb{B}_{M\mu} &= -i\phi(\Gamma_M\Theta)_{\mu}, \\ \mathbb{B}_{\mu\nu} &= 0. \end{aligned} \quad (6.8)$$

In addition, we rewrite (5.35)

$$\phi|_{\Theta=0} = RV^{-1/3}. \quad (6.9)$$

Now consider the two-form $\underline{\mathbb{D}}_{\underline{\mathbb{R}}\underline{\mathbb{S}}}$ associated with the five-brane. Much of the required information can be obtained from the global supersymmetry transformation, which can be read off from the five-brane action after choosing the static gauge. The result is

$$\begin{aligned} \underline{\mathbb{D}}_{\underline{rs}} &= \underline{D}_{\underline{rs}} - \bar{\mathcal{X}}\Gamma_{\underline{rs}}\Theta \\ \underline{\mathbb{D}}_{\underline{r}\underline{\mu}} &= 0 \\ \underline{\mathbb{D}}_{\underline{\mu}\underline{\nu}} &= 0, \end{aligned} \quad (6.10)$$

where \mathcal{X} is the 32-component spinor satisfying $(1 - \Gamma_{11})\mathcal{X} = (1 + i\Gamma_{012345})\mathcal{X} = 0$. Its eight independent components form the spinor χ of the $(2, 0)$ tensor multiplet on the five-brane worldvolume \underline{M}_6 . Finally, we find that

$$\frac{Y}{\pi\rho}\phi = \frac{Y}{\pi\rho}RV^{-1/3} + \bar{\mathcal{X}}\Theta, \quad (6.11)$$

where we have used (6.9). At this point, motivated by the formalism in [28], we make the field redefinition

$$\mathcal{X} = RV^{-1/3} \mathcal{X}_Y. \quad (6.12)$$

Expression (6.11) can then be written as

$$\frac{Y}{\pi\rho} \phi = RV^{-1/3} \mathbb{Y}, \quad (6.13)$$

where

$$\mathbb{Y} = \frac{Y}{\pi\rho} + \bar{\mathcal{X}}_Y \Theta. \quad (6.14)$$

Hence, fermion \mathcal{X}_Y is directly related to the pure translation modulus Y . Substituting these expressions into actions (6.3) and (6.4), they can be written as

$$S_S + S_5 = S_0 + S_\Theta + S_{\Theta^2}, \quad (6.15)$$

where S_0 is purely bosonic

$$\begin{aligned} S_0(X; E_M^A(X), \underline{D}_{rs}(X)) &= T_S \frac{Y}{\pi\rho} \int_C d^2\sigma (RV^{-1/3} \sqrt{\det \partial_i X^M \partial_j X^N E_M^A E_N^B \eta_{AB}} \\ &\quad - \frac{i}{2} \varepsilon^{ij} \partial_i X^M \partial_j X^N B_{NM}) \\ &\quad - \frac{i}{6} T_S \int_C d^2\sigma \varepsilon^{ij} \partial_i X^r \partial_j X^s \underline{D}_{sr}, \end{aligned} \quad (6.16)$$

and S_Θ and S_{Θ^2} are the first two terms (linear and quadratic) in the Θ expansion. Straight-forward calculation gives⁶

$$\begin{aligned} S_\Theta(X, \Theta; E_M^A(X), \mathcal{X}_Y(X)) &= T_S \int_C d^2\sigma RV^{-1/3} \sqrt{\det \partial_i X^M \partial_j X^N E_M^A E_N^B \eta_{AB}} \\ &\quad \cdot \frac{1}{2} (\bar{\mathcal{X}}_Y \mathcal{V} - \bar{\mathcal{V}} \mathcal{X}_Y) \end{aligned} \quad (6.17)$$

and

$$\begin{aligned} S_{\Theta^2}(X, \Theta; E_M^A(X)) &= T_S \frac{Y}{\pi\rho} \int_C d^2\sigma RV^{-1/3} \sqrt{\det \partial_i X^M \partial_j X^N E_M^A E_N^B \eta_{AB}} \\ &\quad (g^{ij} + i\epsilon^{ij}) \bar{\Theta} \Gamma_i D_j \Theta, \end{aligned} \quad (6.18)$$

where $D_i \Theta$ is the covariant derivative

$$D_i \Theta = \partial_i \Theta + \partial_i X^N \omega_N^{AB} \Gamma_{AB} \Theta, \quad (6.19)$$

⁶In a space with Minkowski signature, where the spinors are Majorana-Weyl, the fermion product would be $\bar{\mathcal{X}}_Y \mathcal{V}$. However, in Euclidean space, the fermions are Weyl spinors only and this product becomes the hermitian sum $\frac{1}{2}(\bar{\mathcal{X}}_Y \mathcal{V} - \bar{\mathcal{V}} \mathcal{X}_Y)$.

Γ_i is the pullback of the eleven-dimensional Dirac matrices

$$\Gamma_i = \partial_i X^M \Gamma_M, \quad (6.20)$$

and \mathcal{V} is the vertex operator for the five-brane fermion \mathcal{X}_Y , given by

$$\mathcal{V} = (1 + \frac{i}{2} \epsilon^{ij} \partial_i X^r \partial_j X^s \Gamma_{rs}) \Theta. \quad (6.21)$$

The symbol ϵ^{ij} is the totally antisymmetric tensor in two-dimensions, given in terms of the numeric ε^{ij} by

$$\epsilon^{ij} = \frac{\varepsilon^{ij}}{\sqrt{\det g_{ij}}}. \quad (6.22)$$

Now consider the expansion of the superfields in S_{WZW} given in (6.5). Here, we need only consider the bosonic part of the expansion

$$\begin{aligned} S_{0WZW}(X, \omega; A_M(X), E_M^A(X)) &= -\frac{1}{8\pi} \int_C d^2\sigma \operatorname{tr} \left[\frac{1}{2} \sqrt{g} g^{ij} (\omega_i - A_i) \cdot (\omega_j - A_j) + i \varepsilon^{ij} \omega_i A_j \right] \\ &\quad + \frac{1}{24\pi} \int_B d^3\hat{\sigma} i \hat{\varepsilon}^{ijk} \Omega_{\hat{k}\hat{j}}(\hat{\omega}), \end{aligned} \quad (6.23)$$

where $A_i(\sigma) = \partial_i X^M A_M(X(\sigma))$ is the bosonic pullback of $\mathbb{A}_{\mathbb{M}}$. For example, the expansion of $\mathbb{A}_{\mathbb{M}}$ to linear order in Θ contains fermions that are not associated with the moduli of interest in this paper. Hence, they can be ignored. Similarly, we can show that all other terms in the Θ expansion of S_{WZW} are irrelevant to the problem at hand.

Note that, in terms of the coordinate fields X and Θ , the path integral measure in (6.1) becomes⁷

$$\mathcal{D}\mathbb{Z}\mathcal{D}\omega = \mathcal{D}X\mathcal{D}\Theta\mathcal{D}\omega. \quad (6.24)$$

We can now rewrite the two-point function as

$$\begin{aligned} \langle \lambda^I(y_1^u) \lambda^J(y_2^u) \rangle &\propto \int \mathcal{D}\lambda_{\mathbf{Y}} e^{-\int d^4y \lambda_{\mathbf{Y}} \bar{\partial} \lambda_{\mathbf{Y}}} \lambda_{\mathbf{Y}}(y_1^u) \lambda_{\mathbf{Y}}(y_2^u) \\ &\quad \cdot \int \mathcal{D}X \mathcal{D}\Theta e^{-(S_0 + S_{\Theta} + S_{\Theta^2})} \cdot \int \mathcal{D}\omega e^{-S_{0WZW}}. \end{aligned} \quad (6.25)$$

The last factor

$$\int \mathcal{D}\omega e^{-S_{0WZW}} \quad (6.26)$$

behaves somewhat differently and will be discussed in the next section. Here, we simply note that it does not contain the fermion $\lambda_{\mathbf{Y}}$ and, hence, only contributes an overall determinant to the superpotential. This determinant, although physically important, does not affect the rest of the calculation, to which we now turn. To perform the X, Θ path integral, it is essential that we fix any residual gauge freedom in the X and Θ fields.

⁷Since we are working in Euclidean space, the spinor fields Θ are complex. To be consistent, one must use the integration measure $\mathcal{D}\bar{\Theta}\mathcal{D}\Theta$. In this paper, we write the integration measure $\mathcal{D}\Theta$ as a shorthand for $\mathcal{D}\bar{\Theta}\mathcal{D}\Theta$.

Fixing the X and Θ Gauge:

As stated at the beginning of the last section, we have, for simplicity, thus far taken the superembedding coordinates to be $\mathbb{Z} = (X, \Theta)$ where $(1 - \Gamma_{11})\Theta = 0$. Henceforth, however, we must impose the required restrictions of X to ξ and Θ to satisfy $(1 + i\Gamma_{012345})\Theta = 0$. In addition, we will also impose a further choice of gauge. We begin by considering the bosonic coordinates. As discussed in Section 3, we must take all values of X^M to vanish with the exception of

$$X^{\underline{r}}(\sigma) = \xi^{\underline{r}}(\sigma), \quad \underline{r} = 0, 1, \dots, 5. \quad (6.27)$$

Having done this, it is convenient to fix the gauge of the non-vanishing bosonic coordinates by identifying

$$X^{\underline{r}'}(\sigma) = \delta_i^{\underline{r}'} \sigma^i, \quad (6.28)$$

where $\underline{r}' = 0, 1$. This choice, which corresponds to orienting the X^0 and X^1 coordinates along the string worldvolume, can always be imposed. This leaves four real bosonic degrees of freedom, which we denote as

$$X^u(\sigma) \equiv y^u(\sigma), \quad (6.29)$$

where $u = 2, \dots, 5$. Now consider the fermionic coordinate fields Θ . First make an two-eight split in the Dirac matrices

$$\Gamma_A = (\tau_{a'} \otimes \tilde{\gamma}, 1 \otimes \gamma_{a''}), \quad (6.30)$$

where $a' = 0, 1$ and $a'' = 2, \dots, 9$ are flat indices and $\tau_{a'}$ and $\gamma_{a''}$ are the two- and eight-dimensional Dirac matrices, respectively. Then $\Gamma_{11} \equiv -i\Gamma_0\Gamma_1\cdots\Gamma_9$ can be decomposed as

$$\Gamma_{11} = \tilde{\tau} \otimes \tilde{\gamma} \quad (6.31)$$

where $\tilde{\gamma} = \gamma_2\gamma_3\cdots\gamma_9$ and

$$\tilde{\tau} = -i\tau_0\tau_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (6.32)$$

More explicitly,

$$\Gamma_{11} = \begin{pmatrix} \tilde{\gamma} & 0 \\ 0 & -\tilde{\gamma} \end{pmatrix}. \quad (6.33)$$

Also note that

$$-i\Gamma_{012345} = \begin{pmatrix} \gamma_{2345} & 0 \\ 0 & -\gamma_{2345} \end{pmatrix}. \quad (6.34)$$

In general, the Weyl spinor Θ can be written in a generic basis as

$$\Theta = \begin{pmatrix} \Theta_1 \\ \Theta_2 \end{pmatrix}. \quad (6.35)$$

However, as discussed previously, Θ satisfies

$$\hat{\Gamma}_{\hat{1}\hat{1}}\Theta = \Theta \quad \text{and} \quad i\Gamma_{012345}\Theta = \Theta, \quad (6.36)$$

so that the first condition implies

$$\tilde{\gamma}\Theta_1 = \Theta_1, \quad \tilde{\gamma}\Theta_2 = -\Theta_2, \quad (6.37)$$

and the second one gives

$$\gamma_{2345}\Theta_1 = \Theta_1, \quad \gamma_{2345}\Theta_2 = -\Theta_2 \quad (6.38)$$

From the first equation of (6.36), we conclude that Θ is in the representation $\mathbf{16}^+$ of $SO(10)$. In the presence of the five-brane, $SO(10)$ is broken to $SO(4) \times SO(6) \approx SU(2) \times SU(2) \times SO(6)$ under which

$$\mathbf{16}^+ = (\mathbf{2}^+, \mathbf{1}, \mathbf{4}^+) \oplus (\mathbf{1}, \mathbf{2}^-, \mathbf{4}^-). \quad (6.39)$$

The second projection in (6.36) then implies that Θ is in the representation $(\mathbf{2}^+, \mathbf{1}, \mathbf{4}^+)$. Here, the \pm on $\mathbf{2}$ denote $SO(4)$ chirality and the \pm on $\mathbf{4}$ denote $SO(6)$ chirality. Under the bosonic gauge fixing $X^0 = \sigma^0$ and $X^1 = \sigma^1$, $SO(6)$ is reduced to $SO(4) \times SO(2)$, for which

$$\mathbf{4}^+ = \mathbf{2}^+ \otimes \mathbf{1}^+ \oplus \mathbf{2}^- \otimes \mathbf{1}^-. \quad (6.40)$$

Having applied all the chirality constraints, we can now discuss the decomposition of Θ under the fermionic gauge fixing conditions.

Recall from our discussion of κ -symmetry in Section 2 that, because we can use the κ -invariance of the worldvolume theory to gauge away half of the independent components of Θ , only half of these components represent physical degrees of freedom. For the superstring in Euclidean space, we can define the projection operators

$$P_{\pm} = \frac{1}{2}(1 \pm \frac{i}{2\sqrt{g}}\varepsilon^{ij}\Pi_i^A\Pi_j^B\Gamma_{AB}) \quad (6.41)$$

and write

$$\Theta = P_+\Theta + P_-\Theta. \quad (6.42)$$

Now, note from (2.7) that $P_+\Theta$ can be gauged away while the physical degrees of freedom are given by $P_-\Theta$. Using (6.32), it follows that Θ_2 in (6.35) can be gauged to zero, leaving

only Θ_1 as the physical degrees of freedom. We thus can fix the fermion gauge so that

$$\Theta = \begin{pmatrix} \theta \\ 0 \end{pmatrix}, \quad (6.43)$$

where θ satisfies

$$\tilde{\gamma}\theta = \theta, \quad \gamma_{2345}\theta = \theta, \quad (6.44)$$

and transforms in the representation

$$(\mathbf{2}^+, \mathbf{1}, \mathbf{2}^+, \mathbf{1}^+) \quad (6.45)$$

under $SU(2) \times SU(2) \times SO(4) \times SO(2)$. This corresponds to choosing $\mathbf{1}^+$ under the $SO(2)$ chirality of the string worldsheet, which implies that the physical Θ is the right-moving mode. We conclude that the physical degrees of freedom contained in $\mathbb{Z} = (X, \Theta)$ are

$$y^u(\sigma), \quad \theta_\alpha^A(\sigma), \quad (6.46)$$

where $u = 2, \dots, 5$ indexes R_4 , $A = 1, 2$ is the $SU(2)$ index and α denotes the $\mathbf{2}^+$ of the $SO(4)$ symmetry of R_4 . Therefore, the X, Θ path-integral measures in (6.25) must be rewritten as

$$\mathcal{D}X\mathcal{D}\Theta \propto \mathcal{D}y\mathcal{D}\theta, \quad (6.47)$$

where there is an unimportant constant of proportionality representing the original gauge redundancy.⁸

Equations of Motion:

We can now rewrite the two-point function (6.25) as

$$\begin{aligned} \langle \lambda_{\mathbf{Y}}(y_1^u) \lambda_{\mathbf{Y}}(y_2^u) \rangle &\propto \int \mathcal{D}\lambda_{\mathbf{Y}} e^{-\int d^4y \lambda_{\mathbf{Y}} \bar{\theta}^{\lambda_{\mathbf{Y}}} \lambda_{\mathbf{Y}}} \lambda_{\mathbf{Y}}(y_1^u) \lambda_{\mathbf{Y}}(y_2^u) \\ &\cdot \int \mathcal{D}y \mathcal{D}\theta e^{-(S_0 + S_\Theta + S_{\Theta^2})} \cdot \int \mathcal{D}\omega e^{-S_{0WZW}}. \end{aligned} \quad (6.48)$$

In this paper, we want to use a saddle-point approximation to evaluate these path-integrals. We will consider small fluctuations δy and $\delta\theta$ of the superstring degrees of freedom around a solution y_0 and θ_0 to the equations of motion

$$y = y_0 + \delta y, \quad \theta = \theta_0 + \delta\theta. \quad (6.49)$$

However, before expanding the action using (6.49), we need to discuss the equations of motion for the fields y and θ , as well as their zero-modes.

⁸Here, again, we write $\mathcal{D}\theta$ as a shorthand for $\mathcal{D}\bar{\theta}\mathcal{D}\theta$.

Consider first the equations of motion for the bosonic fields $y(\sigma)$. The bosonic action (6.16) can be written as

$$S_0 = T_S \frac{Y}{\pi \rho} \int_{\mathcal{C}} d^2 \sigma (RV^{-1/3} \sqrt{\det g_{ij}} + \frac{i}{2} \varepsilon^{ij} b_{ij}) - T_S \int_{\mathcal{C}} d^2 \sigma \frac{i}{6} \varepsilon^{ij} \underline{d}_{ij}, \quad (6.50)$$

where

$$g_{ij} = \partial_i X^r \partial_j X^s g_{rs}, \quad b_{ij} = \partial_i X^r \partial_j X^s B_{rs}, \quad \underline{d}_{ij} = \partial_i X^r \partial_j X^s \underline{D}_{rs}. \quad (6.51)$$

We now assume that the background two-form field $B_{MN}(X)$ satisfies $dB = 0$. This can be done if we neglect corrections of order α' . Then, locally, $B = d\Lambda$, where Λ is a one-form. Thus, the second term in (6.50) can be written as a total derivative and so does not contribute to the equations of motion. Next, note that, similarly, the five-brane two-form $\underline{D}_{rs}(X)$ satisfies $d\underline{D} = 0$. This can be seen as follows. Recall from (2.21) that $(d\underline{D})_{rst} = \underline{\mathbb{C}}_{rst}$. However, the field components \mathbb{C}_{MNP} vanish in the low energy limit of heterotic M -theory because of their \mathbb{Z}_2 properties. The result then follows. Therefore, locally, $\underline{D} = d\underline{\Lambda}$, where $\underline{\Lambda}$ is a one-form. Hence, the third term in (6.50) can also be written as a total derivative and so does not contribute to the equations of motion. Varying the action, we obtain the bosonic equations of motion

$$\begin{aligned} \frac{1}{2} \sqrt{\det g_{ij}} g^{kl} \partial_k X^r \partial_l X^s \frac{\partial g_{rs}}{\partial X^t} \\ - \partial_k (\sqrt{\det g_{ij}} g^{kl} \partial_l X^r g_{rt}) = 0, \end{aligned} \quad (6.52)$$

where g^{ij} is the inverse of the induced metric g_{ij} , $g^{ij} g_{jk} = \delta_k^i$. Now fix the bosonic gauge (6.28) and choose a system of coordinates such that the metric tensor restricted to the holomorphic curve \mathcal{C} can be written locally as

$$g_{rs}|_{\mathcal{C}} = \begin{pmatrix} h_{r's'}(\sigma) & 0 \\ 0 & \eta_{uv} \end{pmatrix}, \quad (6.53)$$

where η_{uv} is the flat metric of R_4 . Then equation (6.52) becomes

$$\partial_k \left(\sqrt{\det g_{ij}} \delta_{r'}^k \delta_{s'}^l h_{r's'} \partial_l y_0^u \right) = 0. \quad (6.54)$$

Next, consider the equations of motion for the fermionic degrees of freedom. In action (6.3) the terms that contain Θ are (6.17) and (6.18), whose sum can be written as

$$2T_S \frac{Y}{\pi \rho} \int_{\mathcal{C}} d^2 \sigma RV^{-1/3} \sqrt{\det g_{ij}} \bar{\Theta} \Gamma^i D_i \Theta + \frac{1}{2} T_S \int_{\mathcal{C}} d^2 \sigma RV^{-1/3} \sqrt{\det g_{ij}} (\bar{\mathcal{X}}_Y \mathcal{V} - \bar{\mathcal{V}} \mathcal{X}_Y), \quad (6.55)$$

where we have fixed the gauge as in (6.28) and (6.43), so that \mathcal{V} is given by

$$\begin{aligned} \mathcal{V} &= (1 + \frac{i}{2} \varepsilon^{ij} \partial_i X^r \partial_j X^s \Gamma_{rs}) \Theta \\ &= 2\Theta \end{aligned} \quad (6.56)$$

It follows from the gauge fixing condition (6.43) that only half of the eight independent components of the five-brane fermion \mathcal{X}_Y couple to the physical degrees of freedom in Θ , namely

$$P_+ \mathcal{X}_Y = \frac{1}{2}(1 + i\Gamma_{01})\mathcal{X}_Y \equiv \mathcal{X}_Y^+. \quad (6.57)$$

In fact, the equations of motion for Θ are given by

$$2\frac{Y}{\pi\rho}\Gamma^i D_{0i}\Theta_0 = \mathcal{X}_Y^+, \quad (6.58)$$

where we have used (6.30) and

$$D_{0i}\Theta_0 = \partial_i\Theta_0 + \delta_i^{r'}\omega_{r'}^{\check{K}\check{L}}\Gamma_{\check{K}\check{L}}\Theta_0. \quad (6.59)$$

Of course, we must consider only the physical degrees of freedom θ_0 in Θ_0 .

Zero-Modes:

The saddle-point calculation of the path-integrals $\mathcal{D}y$ and $\mathcal{D}\theta$ around a solution to the equations of motion can be complicated by the occurrence of zero-modes. First consider bosonic solutions $y_0^u(\sigma)$, $u = 2, \dots, 5$ of the equations of motion (6.54). By construction, all such solutions are maps from a holomorphic curve \mathcal{C} to R^4 . Clearly, these can take any value in R_4 , so we can write

$$y_0^u \equiv x^u, \quad (6.60)$$

where x^u are coordinates of R_4 . Therefore, any solution $y_0^u(\sigma)$ of the equations of motion will always have these four translational zero-modes. Are additional zero-modes possible? To avoid this possibility, we will assume in this paper that

$$\mathcal{C} = \mathbb{CP}^1 = S^2, \quad (6.61)$$

where S^2 are rigid spheres isolated in CY_3 . It follows that for a saddle-point calculation of the path-integrals around a rigid, isolated sphere, the bosonic measure can be written as

$$\mathcal{D}y^u = d^4x \mathcal{D}\delta y^u, \quad (6.62)$$

where we have expanded

$$y^u = y_0^u + \delta y^u \quad (6.63)$$

for small fluctuations δy^u .

Now consider fermionic solutions θ_0 of the equation of motion (6.58). To any Θ_0 can always be added a solution of the homogeneous six-dimensional Dirac equation

$$\Gamma^i D_{0i}\Theta' = 0. \quad (6.64)$$

This equation has the general solution

$$\Theta' = \vartheta \otimes \eta_-, \quad (6.65)$$

where η_- is the covariantly constant spinor on CY_3 , which is broken by the embedding of the membrane as discussed in [25], restricted to \mathcal{C} and ϑ is an arbitrary Weyl spinor satisfying the Weyl equation in R_4 . Note that ϑ has negative four-dimensional chirality, since Θ' satisfies $(1 - \Gamma_{11})\Theta' = 0$. Therefore, any solution θ_0 of the equations of motion will always have two complex component fermion zero-modes ϑ^α , $\alpha = 1, 2$. The rigid, isolated sphere has no additional fermion zero-modes. Hence, for a saddle-point calculation of the path integrals around a rigid, isolated sphere the fermionic measure can be written as

$$\mathcal{D}\theta = d\vartheta^1 d\vartheta^2 \mathcal{D}\delta\theta, \quad (6.66)$$

where we have expanded

$$\theta = \theta_0 + \delta\theta \quad (6.67)$$

for small fluctuations $\delta\theta$. To conclude, in the saddle-point approximation the y, θ part of the path integral measure can be written as

$$\mathcal{D}y^u \mathcal{D}\theta = d^4x d\vartheta^1 d\vartheta^2 \mathcal{D}\delta y^u \mathcal{D}\delta\theta. \quad (6.68)$$

Saddle-Point Calculation:

We are now ready to calculate the two-point function (6.48), which can be rewritten as

$$\begin{aligned} \langle \lambda_{\mathbf{Y}}(y_1^u) \lambda_{\mathbf{Y}}(y_2^u) \rangle &\propto \int \mathcal{D}\lambda_{\mathbf{Y}} e^{-\int d^4y \lambda_{\mathbf{Y}} \not{\partial} \lambda_{\mathbf{Y}}} \lambda_{\mathbf{Y}}(y_1^u) \lambda_{\mathbf{Y}}(y_2^u) \\ &\cdot \int d^4x d\vartheta^1 d\vartheta^2 \mathcal{D}\delta y^u \mathcal{D}\delta\theta e^{-(S_0 + S_\Theta + S_{\Theta^2})} \\ &\cdot \int \mathcal{D}\omega e^{-S_{0WZW}}. \end{aligned} \quad (6.69)$$

Substituting the fluctuations (6.49) around the solutions y_0 and θ_0 into

$$\mathcal{S} = S_0 + S_\Theta + S_{\Theta^2}, \quad (6.70)$$

we obtain the expansion

$$\mathcal{S} = \mathcal{S}_0 + \mathcal{S}_2, \quad (6.71)$$

where, schematically

$$\mathcal{S}_0 = \mathcal{S} |_{y_0, \theta_0} \quad (6.72)$$

and

$$\mathcal{S}_2 = \frac{\delta^2 \mathcal{S}}{\delta y \delta y} \big|_{y_0, \theta_0} (\delta y)^2 + 2 \frac{\delta^2 \mathcal{S}}{\delta y \delta \theta} \big|_{y_0, \theta_0} (\delta y \delta \theta) + \frac{\delta^2 \mathcal{S}}{\delta \theta \delta \theta} \big|_{y_0, \theta_0} (\delta \theta)^2. \quad (6.73)$$

The terms in the expansion linear in δy and $\delta \theta$ each vanish by the equations of motion. To avoid further complicating our notation, we state in advance the following simplifying facts. First, note that all terms in \mathcal{S}_2 contribute to the two-point function to order α' on the superstring worldsheet. Therefore, we should evaluate these terms only to classical order in y_0^u and θ_0 . To classical order, one can take $\theta_0 = 0$ since, to this order, the background \mathcal{X}_Y^- field on the right-hand side of (6.58) vanishes. Therefore, \mathcal{S}_2 simplifies to

$$\mathcal{S}_2 = \frac{\delta^2 \mathcal{S}}{\delta y \delta y} \big|_{y_0, \theta_0=0} (\delta y)^2 + \frac{\delta^2 \mathcal{S}}{\delta \theta \delta \theta} \big|_{y_0, \theta_0=0} (\delta \theta)^2. \quad (6.74)$$

It is useful to further denote

$$\mathcal{S}_0 = \mathcal{S}_0^y + \mathcal{S}_0^\theta, \quad (6.75)$$

where

$$\mathcal{S}_0^y = (S_0) \big|_{y_0}, \quad \mathcal{S}_0^\theta = (S_\Theta + S_{\Theta^2}) \big|_{y_0, \theta_0}, \quad (6.76)$$

and to write

$$\mathcal{S}_2 = \mathcal{S}_2^y + \mathcal{S}_2^\theta, \quad (6.77)$$

with

$$\mathcal{S}_2^y = \frac{\delta^2 \mathcal{S}}{\delta y \delta y} \big|_{y_0, \theta_0=0} (\delta y)^2, \quad \mathcal{S}_2^\theta = \frac{\delta^2 \mathcal{S}}{\delta \theta \delta \theta} \big|_{y_0, \theta_0=0} (\delta \theta)^2. \quad (6.78)$$

We can then rewrite two-point function (6.69) as

$$\begin{aligned} \langle \lambda_Y(y_1^u) \lambda_Y(y_2^u) \rangle &\propto \int \mathcal{D}\lambda_Y e^{-\int d^4 y \lambda_Y \not{\partial} \lambda_Y} \lambda_Y(y_1^u) \lambda_Y(y_2^u) \\ &\cdot \int d^4 x e^{-\mathcal{S}_0^y} \cdot \int d\vartheta^1 d\vartheta^2 e^{-\mathcal{S}_0^\theta} \\ &\cdot \int \mathcal{D}\delta y^u e^{-\mathcal{S}_2^y} \cdot \int \mathcal{D}\delta \theta e^{-\mathcal{S}_2^\theta} \cdot \int \mathcal{D}\omega e^{-S_{0WZW}}. \end{aligned} \quad (6.79)$$

We will now evaluate each of the path-integral factors in this expression one by one. We begin with $\int d^4 x e^{-\mathcal{S}_0^y}$.

The \mathcal{S}_0^y Term:

It follows from (6.76) that \mathcal{S}_0^y is simply S_0 , given in (6.50) and (6.51), evaluated at a solution of the equations of motion y_0^u . That is

$$\mathcal{S}_0^y = T_S \frac{Y}{\pi \rho} \int_C d^2 \sigma (R V^{-1/3} \sqrt{\det g_{ij}} + \frac{i}{2} \varepsilon^{ij} b_{ij}) + \frac{i}{6} T_S \int_C d^2 \sigma \varepsilon^{ij} \underline{d}_{ij}, \quad (6.80)$$

where

$$g_{ij} = \partial_i y_0^r \partial_j y_0^s g_{rs}, \quad b_{ij} = \partial_i y_0^r \partial_j y_0^s B_{rs}, \quad \underline{d}_{ij} = \partial_i y_0^r \partial_j y_0^s \underline{D}_{rs}. \quad (6.81)$$

Let us evaluate the term involving g_{ij} . To begin, we note that

$$\int_{\mathcal{C}} d^2 \sigma \sqrt{\det g_{ij}} = \frac{1}{2} \int_{\mathcal{C}} d^2 \sigma \sqrt{g} g^{ij} \partial_i y_0^r \partial_j y_0^s g_{rs}, \quad (6.82)$$

where the first term is obtained from the second using the worldvolume metric equation of motion. Noting that g_{ij} is conformally flat and going to complex coordinates $z = \sigma^0 + i\sigma^1$, $\bar{z} = \sigma^0 - i\sigma^1$, it follows from (6.82) that

$$\int_{\mathcal{C}} d^2 \sigma \sqrt{\det g_{ij}} = \frac{1}{2} \int_{\mathcal{C}} d^2 z \partial_z y_0^r \partial_{\bar{z}} y_0^s \omega_{rs} = \frac{1}{2} \int_{\mathcal{C}} d^2 z \omega_{z\bar{z}}, \quad (6.83)$$

where $\omega_{rs} = i g_{rs}$ is the Kähler form restricted to \mathcal{C} . Using the expansion (5.15) and the orthonormal conditions (5.12), (5.13), it follows from (6.83) that

$$\int_{\mathcal{C}} d^2 \sigma R V^{-1/3} \sqrt{\det g_{ij}} = \frac{v_{\mathcal{C}}}{2} \text{Re} \mathcal{T}. \quad (6.84)$$

Next consider the second term in (6.80) involving b_{ij} . Note that

$$\frac{i}{2} \int_{\mathcal{C}} d^2 \sigma \varepsilon^{ij} b_{ij} = \frac{i}{2} \int_{\mathcal{C}} d^2 z \partial_z y_0^r \partial_{\bar{z}} y_0^s B_{rs} = \frac{i}{2} \int_{\mathcal{C}} d^2 z B_{z\bar{z}}. \quad (6.85)$$

Recall from (5.18) that

$$B_{z\bar{z}} = \text{Im} \mathcal{T} \omega_{\mathcal{C} z \bar{z}} + \dots, \quad (6.86)$$

where the dots indicate terms that vanish upon integration over \mathcal{C} . It follows from (5.12) and (6.85) that

$$\frac{i}{2} \int_{\mathcal{C}} d^2 \sigma \varepsilon^{ij} b_{ij} = \frac{i}{2} v_{\mathcal{C}} \text{Im} \mathcal{T}. \quad (6.87)$$

Finally, consider the third term in (6.80) involving \underline{d}_{ij} . First, we note that

$$\frac{i}{6} \int_{\mathcal{C}} d^2 \sigma \varepsilon^{ij} \underline{d}_{ij} = \frac{i}{6} \int_{\mathcal{C}} d^2 z \partial_z y_0^r \partial_{\bar{z}} y_0^s \underline{D}_{rs} = \frac{i}{6} \int_{\mathcal{C}} d^2 z \underline{D}_{z\bar{z}}. \quad (6.88)$$

Remembering from (5.22) that

$$\underline{D}_{z\bar{z}} = 3a \omega_{\mathcal{C} z \bar{z}}, \quad (6.89)$$

it follows from (5.12) and (6.88) that

$$\frac{i}{6} \int_{\mathcal{C}} d^2 \sigma \varepsilon^{ij} \underline{d}_{ij} = \frac{i}{2} v_{\mathcal{C}} a. \quad (6.90)$$

Putting (6.84), (6.87) and (6.90) together in (6.80), we see that

$$\mathcal{S}_0^y = \frac{T}{2} \left(\frac{Y}{\pi \rho} \text{Re} \mathcal{T} + i \left(a + \frac{Y}{\pi \rho} \text{Im} \mathcal{T} \right) \right), \quad (6.91)$$

where

$$T = T_S v_{\mathcal{C}} = T_M \pi \rho v_{\mathcal{C}} \quad (6.92)$$

is a dimensionless parameter. Recalling from (5.23) that the \mathbf{Y} modulus is defined by

$$\mathbf{Y} = \frac{Y}{\pi \rho} \text{Re} \mathcal{T} + i(a + \frac{Y}{\pi \rho} \text{Im} \mathcal{T}), \quad (6.93)$$

it follows that we can write \mathcal{S}_0^y as

$$\mathcal{S}_0^y = \frac{T}{2} \mathbf{Y}. \quad (6.94)$$

We conclude that the $\int d^4x e^{-\mathcal{S}_0^y}$ factor in the path-integral is given by

$$\int d^4x e^{-\mathcal{S}_0^y} = \int d^4x e^{-\frac{T}{2} \mathbf{Y}}. \quad (6.95)$$

We next evaluate the path integral factor $\int d\vartheta^1 d\vartheta^2 e^{-\mathcal{S}_0^\theta}$.

The \mathcal{S}_0^θ Term and the Fermionic Zero-Mode Integral:

It follows from (6.76) that \mathcal{S}_0^θ is the sum of S_Θ and \mathcal{S}_{Θ^2} , given in (6.55), evaluated at a solution of the equations of motion y_0^u, θ_0 . Varying (6.55) with respect to $\bar{\Theta}$ leads to the equation of motion (6.58). Inserting the equation of motion into (6.55), we find

$$\mathcal{S}_0^\theta = T_S \int_{\mathcal{C}} d^2\sigma R V^{-1/3} \sqrt{\det g_{ij}} \bar{\mathcal{X}}_Y \Theta_0. \quad (6.96)$$

As discussed above, any solution Θ_0 can be written as the sum

$$\Theta_0 = \hat{\Theta}_0 + \Theta', \quad (6.97)$$

where Θ' is a solution of the purely homogeneous Dirac equation (6.64) and has the form (6.65). Since, in the path-integral, we must integrate over the two zero-modes ϑ^α , $\alpha = 1, 2$ in Θ' , it follows that terms involving $\hat{\Theta}_0$ can never contribute to the fermion two-point function. Therefore, when computing the superpotential, one can simply drop $\hat{\Theta}_0$. Hence, \mathcal{S}_0^θ is given by (6.96) where Θ_0 is replaced by Θ' .

Next, we note that the Kaluza-Klein Ansatz for the ten-dimensional fermion \mathcal{X}_Y is given by

$$\mathcal{X}_Y = -i \lambda_Y \otimes \eta_-, \quad (6.98)$$

where $\lambda_Y(y^u)$ are the fermionic superpartners of the complex modulus Y with four-dimensional negative chirality. Using (6.65) and (6.98), one can evaluate the product $\bar{\mathcal{X}}_Y \Theta'$, which is found to be

$$\bar{\mathcal{X}}_Y \Theta' = -i \cdot (\lambda_Y \vartheta), \quad (6.99)$$

where $\lambda_Y \vartheta = \lambda_{Y\alpha} \vartheta^\alpha$ and we used the fact that the CY_3 covariantly constant spinor η_- is normalized to one. Substituting this expression into (6.96) and using (6.84) then gives

$$\mathcal{S}_0^\theta = T \operatorname{Re} \mathcal{T} \lambda_Y \vartheta. \quad (6.100)$$

However, we are not quite finished. Thus far, in this section, we have ignored the gravitino for notational simplicity and because we have presented the gravitino formalism in detail in [25]. Using that formalism, it is straightforward to compute the contribution of the gravitino to \mathcal{S}_0^θ , which we find to be

$$T \frac{Y}{\pi \rho} \lambda_{\mathcal{T}} \vartheta, \quad (6.101)$$

where $\lambda_{\mathcal{T}}$ is the fermionic superpartner of modulus \mathcal{T} discussed in Section 5. Combining (6.100) with (6.101), we have the complete result that

$$\mathcal{S}_0^\theta = T \lambda_{\mathbf{Y}} \vartheta, \quad (6.102)$$

where

$$\lambda_{\mathbf{Y}} = \operatorname{Re} \mathcal{T} \lambda_Y + Y \lambda_{\mathcal{T}} \quad (6.103)$$

is the fermionic superpartner of modulus \mathbf{Y} . It is gratifying that this expression for $\lambda_{\mathbf{Y}}$, as well as expression (6.93) for \mathbf{Y} , are consistent with those found, in a different context, in [28]. It follows that the $\int d\vartheta^1 d\vartheta^2 e^{-\mathcal{S}_0^\theta}$ factor in the path-integral is

$$\int d\vartheta^1 d\vartheta^2 e^{-\mathcal{S}_0^\theta} = \int d\vartheta^1 d\vartheta^2 e^{-T \lambda_{\mathbf{Y}} \vartheta}. \quad (6.104)$$

Expanding the exponential, and using the properties of the Berezin integrals, we find that

$$\int d\vartheta^1 d\vartheta^2 e^{-\mathcal{S}_0^\theta} = \frac{T^2}{2} \lambda_{\mathbf{Y}} \lambda_{\mathbf{Y}}, \quad (6.105)$$

where we have suppressed the spinor indices on $\lambda_{\mathbf{Y}} \lambda_{\mathbf{Y}}$. Collecting the results we have obtained thus far, two-point function (6.79) can now be written as

$$\begin{aligned} \langle \lambda_{\mathbf{Y}}(y_1^u) \lambda_{\mathbf{Y}}(y_2^u) \rangle &\propto \int \mathcal{D} \lambda_{\mathbf{Y}} e^{-\int d^4 y \lambda_{\mathbf{Y}} \not{\partial} \lambda_{\mathbf{Y}}} \lambda_{\mathbf{Y}}(y_1^u) \lambda_{\mathbf{Y}}(y_2^u) \\ &\cdot \int d^4 x e^{-\frac{T}{2} \mathbf{Y}(x)} \lambda_{\mathbf{Y}}(x) \lambda_{\mathbf{Y}}(x) \\ &\cdot \int \mathcal{D} \delta y^u e^{-\mathcal{S}_2^y} \cdot \int \mathcal{D} \delta \theta e^{-\mathcal{S}_2^\theta} \cdot \int \mathcal{D} \omega e^{-S_{0WZW}}. \end{aligned} \quad (6.106)$$

Next, we evaluate the bosonic path-integral factor $\int \mathcal{D} \delta y^u e^{-\mathcal{S}_2^y}$.

The \mathcal{S}_2^y Quadratic Term:

It follows from (6.78) that \mathcal{S}_2^y is simply the quadratic term in the $y = y_0 + \delta y$ expansion of \mathcal{S}_0 , given in (6.50) and (6.51). Note that $S_\Theta + S_{\Theta^2}$ does not contribute since the second derivative is to be evaluated for $\theta_0 = 0$. Furthermore, since this contribution to the path-integral is already at order α' , \mathcal{S}_0 should be evaluated to lowest order in α' . As discussed above, to lowest order $dB = 0$ and, hence, the b_{ij} term in (6.50) is a total divergence which can be ignored. In addition, as discussed above, $d\mathcal{D} = 0$ and, thus, the \underline{d}_{ij} term in (6.50) is also a total divergence which can be ignored. Performing the expansion in what is left, we find that

$$S_2^y = T_S \frac{Y}{\pi\rho} \int_{\mathcal{C}} d^2\sigma R V^{-1/3} \sqrt{\det g_{ij}} \left(\frac{1}{2} g^{ij} (D_i \delta y^u) (D_j \delta y^v) \eta_{uv} \right). \quad (6.107)$$

The induced covariant derivative of δy is a simple ordinary derivative

$$D_i \delta y^u = \partial_i \delta y^u + \omega_i^u{}_v \delta y^v = \partial_i \delta y^u, \quad (6.108)$$

since the connection components vanish along R^4 . Integrating the derivatives by parts then gives

$$S_2^y = T_S \frac{Y}{\pi\rho} \int_{\mathcal{C}} d^2\sigma R V^{-1/3} \left(-\frac{1}{2} \delta y^u [\eta_{uv} \sqrt{g} g^{ij} \mathcal{D}_i \partial_j] \delta y^v \right) \quad (6.109)$$

where the symbol \mathcal{D}_i indicates the covariant derivative with respect to the worldvolume connection on \mathcal{C} . Generically, the fields R, V and Y are functions of x^u . However, as discussed above, at the level of the quadratic contributions to the path-integrals all terms should be evaluated at the classical values of the background fields. Since R, V and Y are moduli, these classical values can be taken to be constants, rendering $Y R V^{-1/3}$ independent of x^u . Hence, the factor $T_S \frac{Y}{\pi\rho} R V^{-1/3}$ can simply be absorbed by a redefinition of the δy 's.

Using the relation

$$\int \mathcal{D}\delta y e^{-\frac{1}{2} \int d^2\sigma \delta y \mathcal{O} \delta y} \propto \frac{1}{\sqrt{\det \mathcal{O}}}, \quad (6.110)$$

we conclude that

$$\int \mathcal{D}\delta y^u e^{-S_2^y} \propto \frac{1}{\sqrt{\det \mathcal{O}_1}} \quad (6.111)$$

where

$$\mathcal{O}_1 = \eta_{uv} \sqrt{g} g^{ij} \mathcal{D}_i \partial_j \quad (6.112)$$

We next turn to the evaluation of the $\int \mathcal{D}\theta e^{-S_2^\theta}$ factor in the path-integral.

The \mathcal{S}_2^θ Quadratic Term:

It follows from (6.78) that \mathcal{S}_2^θ is the quadratic term in the $\theta = \theta_0 + \delta\theta$ expansion of S_{Θ^2} , given in (6.18). Note that $S_0 + S_\Theta$ does not contribute. Performing the expansion and taking into account the gauge fixing condition, we find that

$$\mathcal{S}_2^\theta = 2T_S \frac{Y}{\pi\rho} \int_{\mathcal{C}} d^2\sigma R V^{-1/3} \sqrt{\det g_{ij}} \delta\bar{\Theta} \Gamma^i D_i \delta\Theta. \quad (6.113)$$

One must now evaluate the product $\delta\bar{\Theta} \Gamma^i D_i \delta\Theta$ in terms of the gauged-fixed quantities $\delta\theta$. We start by rewriting

$$\begin{aligned} \delta\bar{\Theta} \Gamma^i D_i \delta\Theta &= g^{ij} \partial_j X^r \delta\bar{\Theta} \Gamma_r \partial_i \delta\Theta \\ &\quad + g^{ij} \partial_j X^r \partial_i X^s \omega_{\underline{s}}^{AB} \delta\bar{\Theta} \Gamma_r \Gamma_{AB} \delta\Theta, \end{aligned} \quad (6.114)$$

where $A = (a', a'')$ and we have used the restrictions on fields $X^M(\sigma)$. After fixing the gauge freedom of the bosonic fields $X^r(\sigma)$ as in (6.28), expression (6.114) becomes

$$\begin{aligned} \delta\bar{\Theta} \Gamma^i D_i \delta\Theta &= g^{ij} \delta_j^{m'} e_{m'}^{a'} \delta\bar{\Theta} \Gamma_{a'} \partial_i \delta\Theta + g^{ij} e_u^k \partial_j y^u \delta\bar{\Theta} \Gamma_k \partial_i \delta\Theta \\ &\quad + g^{ij} \delta_j^{m'} e_{m'}^{a'} \delta_i^{n'} \omega_{n'}^{AB} \delta\bar{\Theta} \Gamma_{a'} \Gamma_{AB} \delta\Theta \\ &\quad + g^{ij} e_u^k \partial_j y^u \delta_i^{m'} \omega_{m'}^{AB} \delta\bar{\Theta} \Gamma_k \Gamma_{AB} \delta\Theta, \end{aligned} \quad (6.115)$$

where $k = 2, 3, 4, 5$ are flat indices in R_4 . We see that we must evaluate the fermionic products

$$\delta\bar{\Theta} \Gamma_{a'} \partial_i \delta\Theta, \quad \delta\bar{\Theta} \Gamma_k \partial_i \delta\Theta, \quad \delta\bar{\Theta} \Gamma_{a'} \Gamma_{AB} \delta\Theta, \quad \delta\bar{\Theta} \Gamma_k \Gamma_{AB} \delta\Theta \quad (6.116)$$

in terms of $\delta\theta$. After fixing the fermionic gauge according to (6.43), we can compute the relevant terms in the expression (6.115). Consider a product of the type $\delta\bar{\Theta} M \delta\Theta$, where M is a 32×32 matrix-operator,

$$M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}. \quad (6.117)$$

Using (6.30) and (6.43), we have

$$\delta\bar{\Theta} M \delta\Theta = \delta\Theta^\dagger M \delta\Theta = \delta\theta^\dagger M_1 \delta\theta \quad (6.118)$$

Therefore, using (6.30), we have the following results

$$\begin{aligned} \delta\bar{\Theta} \Gamma_{a'} \partial_i \delta\Theta &= 0, \quad \delta\bar{\Theta} \Gamma_{a''} \partial_i \delta\Theta = \delta\theta^\dagger \gamma_{a''} \partial_i \delta\theta, \\ \delta\bar{\Theta} \Gamma_{a'} \Gamma_{b'c'} \delta\Theta &= 0, \quad \delta\bar{\Theta} \Gamma_{a'} \Gamma_{b'a''} \delta\Theta = (\delta_{a'b'} - i\varepsilon_{a'b'}) \delta\theta^\dagger \gamma_{a''} \delta\theta, \\ \delta\bar{\Theta} \Gamma_{a'} \Gamma_{a''b''} \delta\Theta &= 0, \quad \delta\bar{\Theta} \Gamma_{a''} \Gamma_{a'b'} \delta\Theta = -i\varepsilon_{a'b'} \delta\theta^\dagger \gamma_{a''} \delta\theta, \\ \delta\bar{\Theta} \Gamma_{a''} \Gamma_{a'b''} \delta\Theta &= 0, \quad \delta\bar{\Theta} \Gamma_{a''} \Gamma_{b''c''} \delta\Theta = \delta\theta^\dagger \gamma_{a''} \gamma_{b''c''} \delta\theta, \end{aligned} \quad (6.119)$$

with $a'' = (k, K)$, $k = 2, 3, 4, 5$ are flat indices in R_4 and $K = 6, 7, 8, 9$ are flat indices in the supspace $CY_\perp \subset CY_3$ orthogonal to \mathcal{C} . Substituting these expressions into (6.115) yields

$$\begin{aligned} \delta\bar{\Theta}\Gamma^i D_i \delta\Theta &= g^{ij} e_u^k \partial_j y^u [\delta\theta^\dagger \gamma_k \partial_i \delta\theta - i\delta_i^{m'} \omega_{m'}^{a'b'} \varepsilon_{a'b'} \delta\theta^\dagger \gamma_k \delta\theta + \delta_i^{m'} \omega_{m'}^{KL} \delta\theta^\dagger \gamma_k \gamma_{KL} \delta\theta] \\ &\quad + g^{ij} \delta_j^{m'} e_{m'}^{a'} \delta_i^{n'} \omega_{n'}^{b'K} (\delta_{a'b'} - i\varepsilon_{a'b'}) \delta\theta^\dagger \gamma_K \delta\theta. \end{aligned} \quad (6.120)$$

Then (6.113) becomes

$$\begin{aligned} \mathcal{S}_2^\theta &= 2T_S \frac{Y}{\pi\rho} \int_{\mathcal{C}} d^2\sigma R V^{-1/3} \delta\theta^\dagger \{ \sqrt{g} g^{ij} e_u^k \partial_j y^u [\gamma_k \partial_i - i\delta_i^{m'} \omega_{m'}^{a'b'} \varepsilon_{a'b'} \gamma_k \\ &\quad + \delta_i^{m'} \omega_{m'}^{KL} \gamma_k \gamma_{KL}] + \sqrt{g} g^{ij} \delta_j^{m'} e_{m'}^{a'} \delta_i^{n'} \omega_{n'}^{b'K} (\delta_{a'b'} - i\varepsilon_{a'b'}) \gamma_K \} \delta\theta. \end{aligned} \quad (6.121)$$

As discussed in the previous section, at the level of the quadratic contributions to the path-integrals, all terms should be evaluated at the classical values of the background fields. Therefore, the factor $2T_S \frac{Y}{\pi\rho} R V^{-1/3}$ can be absorbed by a redefinition of the $\delta\theta$'s. Next, we use the relation

$$\int \mathcal{D}\delta\theta e^{\int d^2\sigma \delta\theta^\dagger \mathcal{O} \delta\theta} \propto \det \mathcal{O}. \quad (6.122)$$

Note, however, that when going to Euclidean space, we have doubled the number of fermion degrees of freedom. Therefore, one must actually integrate over only one half of these degrees of freedom. This amounts to taking the square-root of the determinant on the right-hand side of (6.122). Hence, we conclude that

$$\int \mathcal{D}\delta\theta e^{-\mathcal{S}_2^\theta} \propto \sqrt{\det \mathcal{Q}_3}, \quad (6.123)$$

where

$$\begin{aligned} \mathcal{Q}_3 &= \sqrt{g} g^{ij} \{ \gamma_k e_u^k \partial_j y^u [\partial_i - i\delta_i^{m'} \omega_{m'}^{a'b'} \varepsilon_{a'b'} + \delta_i^{m'} \omega_{m'}^{KL} \gamma_{KL}] \\ &\quad + \gamma_K \delta_j^{m'} e_{m'}^{a'} \delta_i^{n'} \omega_{n'}^{b'K} (\delta_{a'b'} - i\varepsilon_{a'b'}) \}. \end{aligned} \quad (6.124)$$

Note that because of the projections (6.44) that reduce the number of independent components of θ from 16 to 4, the operator \mathcal{Q}_3 must be projected accordingly. We implicitly assume this.

Collecting the results we have obtained thus far, two-point function (6.79) can now be written as

$$\begin{aligned} \langle \lambda_{\mathbf{Y}}(y_1^u) \lambda_{\mathbf{Y}}(y_2^u) \rangle &\propto \frac{\sqrt{\det \mathcal{Q}_3}}{\sqrt{\det \mathcal{O}_1}} \cdot \int \mathcal{D}\lambda_{\mathbf{Y}} e^{-\int d^4 y \lambda_{\mathbf{Y}} \not{\partial} \lambda_{\mathbf{Y}}} \lambda_{\mathbf{Y}}(y_1^u) \lambda_{\mathbf{Y}}(y_2^u) \\ &\quad \cdot \int d^4 x e^{-\frac{T}{2} \mathbf{Y}} \lambda_{\mathbf{Y}}(x) \lambda_{\mathbf{Y}}(x) \\ &\quad \cdot \int \mathcal{D}\omega e^{-S_{0WZW}}. \end{aligned} \quad (6.125)$$

It remains, therefore, to evaluate the $\int \mathcal{D}\omega e^{-S_{0WZW}}$ factor in the path-integral, which we now turn to.

7 The Wess-Zumino-Witten Determinant:

In this section, we will discuss the E_8 Wess-Zumino-Witten part of the action, its quadratic expansion and one loop determinant. Here we follow the exposition in [25] closely.

Recall from (6.23) that the relevant action is

$$\begin{aligned} S_{0WZW} = & -\frac{1}{8\pi} \int_{\mathcal{C}} d^2\sigma \operatorname{tr} \left[\frac{1}{2} \sqrt{g} g^{ij} (\omega_i - A_i) \cdot (\omega_j - A_j) + i\varepsilon^{ij} \omega_i A_j \right] \\ & + \frac{1}{24\pi} \int_{\mathcal{B}} d^3\hat{\sigma} i\varepsilon^{\hat{i}\hat{j}\hat{k}} \Omega_{\hat{k}\hat{j}\hat{i}}(\hat{\omega}), \end{aligned} \quad (7.1)$$

where $\omega = g^{-1}dg$ is an E_8 Lie algebra valued one-form and g is given in (3.19). In order to discuss the equation of motion and the chirality of this action, it is convenient to use the complex coordinates $z = \sigma^0 + i\sigma^1$, $\bar{z} = \sigma^0 - i\sigma^1$ on \mathcal{C} and to define the complex components of A by $A = A_z dz + A_{\bar{z}} d\bar{z}$. Then action (7.1) can be written as

$$\begin{aligned} S_{0WZW} = & -\frac{1}{8\pi} \int_{\mathcal{C}} d^2z \operatorname{tr} (g^{-1} \partial_z g g^{-1} \partial_{\bar{z}} g - 2A_z g^{-1} \partial_{\bar{z}} g + A_{\bar{z}} A_z) \\ & + \frac{1}{24\pi} \int_{\mathcal{B}} d^3\hat{\sigma} i\varepsilon^{\hat{i}\hat{j}\hat{k}} \Omega_{\hat{k}\hat{j}\hat{i}}(\hat{\omega}). \end{aligned} \quad (7.2)$$

It is useful to define the two E_8 currents

$$J_z = (D_z g) g^{-1}, \quad J_{\bar{z}} = g^{-1} D_{\bar{z}} g, \quad (7.3)$$

where D_z and $D_{\bar{z}}$ are the E_8 covariant derivatives. In order to perform the path-integral over ω , it is necessary to fix any residual gauge freedom in the ω fields. Recall from the discussion in Section 3 that the entire action is invariant under both local gauge and modified κ -transformations, $\delta_{\mathbb{L}}$ and $\Delta_{\hat{k}}$ respectively. It follows from (3.21) and (3.14) that

$$\delta_{\hat{k}} g = g i_{\hat{k}} \mathbb{A}. \quad (7.4)$$

It is not difficult to show that using this transformation, one can choose a gauge where

$$J_z = 0. \quad (7.5)$$

Henceforth, we work in this chiral gauge. It follows from (7.2) that the g equations of motion are

$$\partial_{\bar{z}} J_z = 0, \quad D_z J_{\bar{z}} + F_{z\bar{z}} = 0, \quad (7.6)$$

where $F_{z\bar{z}}$ is the E_8 field strength. Note that this is consistent with the gauge choice (7.5) and, hence, that the first equation in (7.6) is vacuous. Thus, the on-shell theory we obtain from the gauged Wess-Zumino-Witten action is an E_8 chiral current algebra at level one.

The level can be read off from the coefficient of the Chern-Simons term in (7.2). We would now like to evaluate the Wess-Zumino-Witten contribution to the path-integral using a saddle-point approximation. To do this, we should expand g as small fluctuations

$$g = g_0 + \delta g \quad (7.7)$$

around a classical solution g_0 of (7.6). However, it is clearly rather difficult to carry out the quadratic expansion and evaluate the determinant in this formalism. Luckily, there is an equivalent theory which is more tractable in this regard, which we now describe.

As discussed in [25], if the gauge field background is restricted to lie within an $SO(16)$ subgroup of E_8 , then the equivalent action is given by the free fermion theory coupled to the $SO(16)$ gauge field background. As described in [25], realistic heterotic M -theory models can always be chosen to have the gauge instanton within the $SO(16)$ subgroup of E_8 . Here, we consider only such restricted backgrounds. We can now write the action for $SO(16)$ fermions coupled to background gauge fields. It is given by [47]–[51]

$$S_\psi = \int_{\mathcal{C}} d^2\sigma \bar{\psi}^a \mathcal{P}_A^{ab} \psi^b \quad (7.8)$$

where ψ^a denotes the set of $SO(16)$ fermions with $a, = 1, \dots, 16$ and

$$\mathcal{P}_A^{ab} = \sqrt{g} \tau^i (D_i \delta^{ab} - A_i^{ab}) \quad (7.9)$$

is the covariant derivative on \mathcal{C} with A_i^{ab} the set of $SO(16)$ background gauge fields. The matrices τ^i are the Dirac matrices in two-dimensions. It follows from the above discussion that we can write

$$\int \mathcal{D}\omega e^{-S_{0WZW}} \propto \int \mathcal{D}\psi^a e^{-S_\psi}, \quad (7.10)$$

where the gauge fixing of variable ω described by (7.5) is inherent in the ψ^a formalism, as we will discuss below. The equations of motion are given by

$$\mathcal{P}_A^{ab} \psi^b = 0. \quad (7.11)$$

We now expand

$$\psi^a = \psi_0^a + \delta\psi^a \quad (7.12)$$

around a solution ψ_0^a of (7.11) and consider terms in S_ψ up to quadratic order in the fluctuations $\delta\psi^a$. We find that

$$S_\psi = S_{0\psi} + S_{2\psi}, \quad (7.13)$$

where

$$S_{0\psi} = \int_{\mathcal{C}} d^2\sigma \bar{\psi}_0^a \mathcal{P}_A^{ab} \psi_0^b \quad (7.14)$$

and

$$S_{2\psi} = \int_{\mathcal{C}} d^2\sigma \delta\bar{\psi}^a \mathcal{P}_A^{ab} \delta\psi^b. \quad (7.15)$$

The terms linear in $\delta\psi$ vanish by the equations of motion. It follows immediately from (7.11) that $S_{0\psi} = 0$. Then, using (6.122), one finds from (7.15) that

$$\int \mathcal{D}\delta\psi^a e^{-S_\psi} \propto \sqrt{\det \mathcal{P}_A}. \quad (7.16)$$

Note, again, that by going to Euclidean space we have doubled the number of fermionic degrees of freedom. Therefore, one must actually integrate over only one half of these degrees of freedom. This requires the square-root of the determinants to appear in (7.16). It is important to discuss how the chiral gauge fixing condition (7.5) is manifested in the ψ^a formalism. Condition (7.5) imposes the constraint that g couples only to the A_z component of the gauge fields and not to $A_{\bar{z}}$. It follows that in evaluating $\det \mathcal{P}_A$, we should keep only the A_z components of the gauge fields. That is, we should consider the Dirac determinants of $SO(16)$ holomorphic vector bundles on the Riemann surface \mathcal{C} . Gauge fixing condition (7.5) also imposes a constraint on the definition of determinant $\det \mathcal{P}_A$ as follows. Recall that on the Euclidean space \mathcal{C} , each spinor ψ is a complex two-component Weyl spinor

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}. \quad (7.17)$$

Rescaling this basis to

$$\begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \begin{pmatrix} (g_{z\bar{z}})^{-1/4} \tilde{\psi}_+ \\ (g_{z\bar{z}})^{1/4} \tilde{\psi}_- \end{pmatrix} \quad (7.18)$$

and using the standard representation for τ^0, τ^1 then, locally, one can write

$$\mathcal{P}_A = \begin{pmatrix} 0 & D_{-A} \\ D_{+A} & 0 \end{pmatrix}, \quad (7.19)$$

where

$$D_{-A} = (g_{z\bar{z}})^{3/4} \left((g_{z\bar{z}})^{-1/2} \frac{\partial}{\partial \bar{z}} (g_{z\bar{z}})^{1/2} - A_z \right), \quad D_{+A} = (g_{z\bar{z}})^{1/4} \frac{\partial}{\partial \bar{z}}. \quad (7.20)$$

Since the operator \mathcal{P}_A must be Hermitean, it follows that $D_{+A} = D_{-A}^\dagger$. Now, in addition to disallowing any coupling to $A_{\bar{z}}$, gauge condition (7.5) imposes the constraint that

$$\psi_+^a = 0 \quad (7.21)$$

for all $a = 1, \dots, 16$. Then, using the fact that

$$\det \mathcal{P}_A = \sqrt{\det(\mathcal{P}_A)^2} \quad (7.22)$$

and gauge condition (7.21), we see that the proper definition of the determinant is

$$\det \mathcal{D}_A = \sqrt{\det D_{-A}^\dagger D_{-A}}. \quad (7.23)$$

In this paper, it is not necessary to determine the exact value of $\det \mathcal{D}_A$. We need only compute whether it vanishes or is non-zero, and the conditions under which these two possibilities occur. To do this, we must examine the global properties of the holomorphic vector bundle. As we did throughout the paper, we will restrict

$$\mathcal{C} = \mathbb{CP}^1 = S^2. \quad (7.24)$$

With this restriction, the condition for the vanishing of $\det \mathcal{D}_A$ can be given explicitly, as we now show.

It follows from (7.21) that the chiral fermions realizing the $SO(16)$ current algebra are elements of the negative chiral spinor line bundle S_- of the sphere. Note from (7.19) that D_{-A} is the part of the Dirac operator which acts on S_- . With respect to a non-trivial $SO(16)$ holomorphic vector bundle background A , the complete operator we should consider is

$$D_{-A} : S_- \otimes A \rightarrow S_+ \otimes A, \quad (7.25)$$

where S_+ denotes the positive chiral spinor bundle on the sphere. This is the global description of the local D_{-A} operator defined in (7.19) and (7.20). In order to have nonzero determinant $\det \mathcal{D}_A$, it is necessary and sufficient that D_{-A} should not have any zero-modes. This follows from the index theorem which, for $SO(16)$, implies that

$$\text{coker} D_{-A}^\dagger = \ker D_{-A}. \quad (7.26)$$

As was shown in [25], D_{-A} does not have any zero-modes if and only if the restriction of the $SO(16)$ holomorphic vector bundle A to \mathcal{C} is trivial. Therefore, in order to have a non-zero superpotential for the five-brane, we must have a special type of the gauge bundle on the Calabi-Yau threefold. Bundles of this type are straightforward to construct. We will present a number of phenomenologically relevant examples in a forthcoming paper [52].

8 Final Expression for the Superpotential:

We are now, finally, in a position to extract the final form of the non-perturbative superpotential from the fermion two-point function. Combining the results of the previous section

with expression (6.125), we find that

$$\begin{aligned}
\langle \lambda_{\mathbf{Y}}(y_1^u) \lambda_{\mathbf{Y}}(y_2^u) \rangle &\propto \frac{\sqrt{\det \mathcal{O}_3}}{\sqrt{\det \mathcal{O}_1}} \cdot \sqrt{\det \mathcal{D}_A} \\
&\cdot \int \mathcal{D}\lambda_{\mathbf{Y}} e^{-\int d^4 y \lambda_{\mathbf{Y}} \not{\partial} \lambda_{\mathbf{Y}}} \lambda_{\mathbf{Y}}(y_1^u) \lambda_{\mathbf{Y}}(y_2^u) \\
&\cdot \int d^4 x e^{-\frac{T}{2} \mathbf{Y}(x)} \lambda_{\mathbf{Y}}(x) \lambda_{\mathbf{Y}}(x).
\end{aligned} \tag{8.1}$$

Comparing this with the purely holomorphic part of the quadratic fermion term in the four-dimensional effective Lagrangian (5.28)

$$(\partial_{\mathbf{Y}} \partial_{\mathbf{Y}} W) \lambda_{\mathbf{Y}} \lambda_{\mathbf{Y}}, \tag{8.2}$$

we obtain

$$W \propto \frac{\sqrt{\det \mathcal{O}_3}}{\sqrt{\det \mathcal{O}_1}} \cdot \sqrt{\det \mathcal{D}_A} \cdot e^{-\frac{T}{2} \mathbf{Y}}. \tag{8.3}$$

In this expression, the dimensionless field \mathbf{Y} is defined by

$$\mathbf{Y} = \frac{Y}{\pi \rho} \text{Re} \mathcal{T} + i(a + \frac{Y}{\pi \rho} \text{Im} \mathcal{T}), \tag{8.4}$$

where Y and a are the translational and axionic moduli of the five-brane respectively and τ is the complex $(1,1)$ -modulus associated with curve \mathcal{C} . T is a dimensionless parameter given by

$$T = T_M \pi \rho \, v_{\mathcal{C}}, \tag{8.5}$$

with T_M the membrane tension and $\pi \rho$ the S^1/\mathbb{Z}_2 interval length. The operators \mathcal{O}_1 and \mathcal{O}_3 are presented in (6.112) and (6.124), respectively. The operator \mathcal{D}_A and its determinant $\det \mathcal{D}_A$ are defined in (7.9), (7.19), (7.20) and (7.23). This determinant and, hence, the superpotential W will be non-vanishing if and only if the pullback of the associated $SO(16)$ holomorphic vector bundle A to the curve \mathcal{C} is trivial. All the determinants contributing to W are non-negative real numbers. We emphasize that W given in (8.3) is the contribution of open supermembranes wrapped once around $\mathcal{C} \times I$, where $\mathcal{C} = S^2$ is a sphere isolated in the Calabi-Yau threefold CY_3 and $I \subset S^1/\mathbb{Z}_2$. The generalization to supermembranes wrapped once around I but n -times around \mathcal{C} is straightforward. One simply replaces the exponential term in (8.3) by

$$e^{-\frac{nT}{2} \mathbf{Y}}. \tag{8.6}$$

Further generalizations and discussions of the complete open supermembrane contributions to the non-perturbative superpotential in heterotic M -theory will be presented elsewhere [52].

A Notation and Conventions:

We use a notation such that symbols and indices without hats represent fields in the ten-dimensional fixed hyperplanes of Hořava-Witten theory (as well as the two-dimensional heterotic string theory), while hatted indices relate to quantities of eleven-dimensional bulk space (and the three-dimensional open membrane theory). In addition, underlined symbols and indices refer to the five-brane worldvolume.

Bosons:

For example,

$$X^M, \quad M = 0, 1, \dots, 9, \quad \text{and} \quad \hat{X}^{\hat{M}}, \quad \hat{M} = \hat{0}, \hat{1}, \dots, \hat{9}, \hat{11}, \quad (\text{A.1})$$

are, respectively, the coordinates of ten- and eleven-dimensional spacetimes. We do not change notation when switching from Minkowskian signature to Euclidean signature.

Eleven-dimensional space is, by assumption, given by

$$M_{11} = R_4 \times CY_3 \times S^1/\mathbb{Z}_2, \quad (\text{A.2})$$

while the ten-dimensional space obtained by compactifying it on S^1/\mathbb{Z}_2 is, clearly,

$$M_{10} = R_4 \times CY_3. \quad (\text{A.3})$$

The membrane worldvolume Σ is decomposed as

$$\Sigma = \mathcal{C} \times I, \quad (\text{A.4})$$

where the holomorphic curve \mathcal{C} lies within CY_3 and $I \subset S^1/\mathbb{Z}_2$.

The two-dimensional heterotic string theory is represented by fields with worldsheet coordinates σ^i , with $i = 0, 1$. Bosonic indices of ten-dimensional spacetime are split into indices parallel to the worldsheet ($m' = 0, 1$) and indices perpendicular to it ($m'' = 2, \dots, 9$). The space normal to the worldsheet is an eight-dimensional space. Since it is assumed that the worldsheet is wrapped on a curve \mathcal{C} contained in the Calabi-Yau threefold CY_3 , these eight directions $y^{m''}$ can be split in two sets of four. The first set parametrizes the subset $CY_\perp \subset CY_3$ which is normal to curve \mathcal{C} . The coordinates are denoted y^U , $U = 6, 7, 8, 9$. The second set consists of the coordinates y^u , $u = 2, 3, 4, 5$ of R_4 .

The five-brane worldvolume \underline{M}_6 is embedded in M_{10} as

$$M_{10} = CY_\perp \times \underline{M}_6, \quad (\text{A.5})$$

where

$$\underline{M}_6 = R_4 \times \mathcal{C}. \quad (\text{A.6})$$

Indices in the five-brane super-worldvolume are given by $\underline{\mathbb{R}} = (\underline{r}, \underline{\mu})$, with $\underline{r} = 0, 1, \dots, 5$ and $\underline{\mu} = 1, \dots, 16$. Note that one has a $(2, 0)$ -supersymmetry on the five-brane worldvolume.

Coordinates of CY_3 are denoted by

$$\check{y}^{\check{U}} = (X^{\underline{r}'}, y^U), \quad \text{with } \check{U} = 0, 1, 6, 7, 8, 9, \quad \underline{r}' = 0, 1, \quad (\text{A.7})$$

or, using the complex structure notation,

$$\check{y}^m, \quad \check{y}^{\bar{m}}, \quad m = 1, 2, 3, \quad \bar{m} = \bar{1}, \bar{2}, \bar{3}. \quad (\text{A.8})$$

The bosonic indices in (A.1)-(A.8) are coordinate (or “curved”) indices. The corresponding tangent space (or “flat”) indices are given in the following table,

M_{10}	M_{11}	\underline{M}_6	\mathcal{C}	M_\perp	R^4	CY_\perp
M, N	\hat{M}, \hat{N}	$\underline{r}, \underline{s}$	$\underline{r}', \underline{s}'$	m'', n''	u, v	U, V
A, B	\hat{A}, \hat{B}	$\underline{a}, \underline{b}$	$\underline{a}', \underline{b}'$	a'', b''	k, l	K, L

where M_\perp is the subspace of M_{10} perpendicular to \mathcal{C} .

Spinors:

In ten-dimensional spacetime with Euclidean signature, the 32×32 Dirac matrices Γ_A satisfy

$$\{\Gamma_A, \Gamma_B\} = 2\eta_{AB} \quad (\text{A.9})$$

or, with curved indices, (since $\Gamma_A = e_A^M \Gamma_M$)

$$\{\Gamma_M, \Gamma_N\} = 2g_{MN}. \quad (\text{A.10})$$

One defines ten-dimensional chirality projection operators $\frac{1}{2}(1 \pm \Gamma_{11})$, where

$$\Gamma_{11} = -i\Gamma_0\Gamma_1 \cdots \Gamma_9. \quad (\text{A.11})$$

A useful representation for Γ_A is given by the two-eight split

$$\Gamma_A = (\tau_{a'} \otimes \tilde{\gamma}, 1 \otimes \gamma_{a''}), \quad (\text{A.12})$$

where the two-dimensional Dirac matrices τ_0, τ_1 and their product defined by $\tilde{\tau} = -i\tau_0\tau_1$ are explicitly given by

$$\tau_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tilde{\tau} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (\text{A.13})$$

These ten-dimensional Dirac matrices are more explicitly written as

$$\begin{aligned} \Gamma_0 &= \begin{pmatrix} 0 & \tilde{\gamma} \\ \tilde{\gamma} & 0 \end{pmatrix} & \Gamma_1 &= \begin{pmatrix} 0 & -i\tilde{\gamma} \\ i\tilde{\gamma} & 0 \end{pmatrix} \\ \Gamma_{a''} &= \begin{pmatrix} \gamma_{a''} & 0 \\ 0 & \gamma_{a''} \end{pmatrix} & \Gamma_{11} &= \begin{pmatrix} \tilde{\gamma} & 0 \\ 0 & -\tilde{\gamma} \end{pmatrix} \end{aligned} \quad (\text{A.14})$$

where $\gamma_{a''}$ are 16×16 Dirac matrices, and the product

$$\tilde{\gamma} = \gamma_2 \gamma_3 \cdots \gamma_9 \quad (\text{A.15})$$

is used in the definition of eight-dimensional chirality projection operators $\frac{1}{2}(1 \pm \tilde{\gamma})$. Note that $\Gamma_{11}^2 = 1$, $\tilde{\gamma}^2 = 1$, and $\tilde{\tau}^2 = 1$. In eleven-dimensions, the 32×32 Dirac matrices are given by

$$\hat{\Gamma}_{\hat{A}} = \Gamma_{\hat{A}}, \quad (\hat{A} = \hat{0}, \hat{1}, \dots, \hat{9}), \quad \text{and} \quad \hat{\Gamma}_{\hat{11}} = \Gamma_{11}. \quad (\text{A.16})$$

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